Syntactical and Semantical approaches to Generic Multiverse

Toshimichi Usuba(薄葉 季路)

Waseda University

December 7 , 2018
Sendai Logic School 2018

Generic Multivserse

- In 1980's, Woodin introduced a concept of Generic Multiverse:

Definition

A collection $\mathcal{M}$ is a Generic Multiverse (GM) if :

1. $\mathcal{M}$ is a family of models of ZFC.
2. $\mathcal{M}$ is closed under taking ground models and generic extensions.
3. For every universe $M, N \in \mathcal{M}$, M is connected with N.

Woodin's generic multiverse

Definition (Woodin 1980's)

A collection $\mathcal{M}$ is a Woodin's generic multiverse if :

1. $\mathcal{M}$ is a family of countable transitive $\in$-models of ZFC.
2. $\mathcal{M}$ is closed under taking ground models; for every $M \in \mathcal{M}$, if $N \subseteq M$ is a ground model of M then $N \in \mathcal{M}$.
3. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in \mathcal{M}$, and $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.
4. For every $M, N \in \mathcal{M}$, there are finitely many $M_0, \ldots, M_n \in \mathcal{M}$ such that $M_0 = M$, $M_n = N$, and each M_i is a ground model or a generic extension of M_{i+1}.

- Woodin's Generic multiverse can be seen as Kripke frame; a family of possible worlds.
- In the view point of multiverse conception, one can say that:
  - The Continuum Hypothesis is neither TRUE nor FALSE, because there are two worlds $M, N \in \mathcal{M}$ such that CH is true in M but false in N.
  - However, (under certain Large Cardinal Axiom), the regularity properties of the projective sets of the reals is TRUE, because it is true in any worlds of $\mathcal{M}$.

- Woodin's Generic multiverse can be seen as Kripke frame; a family of possible worlds.
- In the view point of multiverse conception, one can say that:
  - The Continuum Hypothesis is neither TRUE nor FALSE, because there are two worlds $M, N \in \mathcal{M}$ such that CH is true in M but false in N.
  - However, (under certain Large Cardinal Axiom), the regularity properties of the projective sets of the reals is TRUE, because it is true in any worlds of $\mathcal{M}$.

Remark

Let $\mathcal{M}$ be a Woodin's generic multiverse.

1. Since each $M \in \mathcal{M}$ is countable, $\mathcal{M}$ satisfies: For every $M \in \mathcal{M}$ and $\mathbb{P} \in M$, there is an $(M, \mathbb{P})$-generic G with $M[G] \in \mathcal{M}$.
2. For a given countable transitive $\in$-model M of ZFC, there is a unique Woodin's generic multiverse $\mathcal{M}$ with $M \in \mathcal{M}$. In this sense, $\mathcal{M}$ is the generic multiverse containing M, or the generic multiverse generated by M.

The construction of Woodin's GM

1. Fix a countable transitive $\in$-model M of ZFC.
2. Let $\mathcal{M}_0 = \{M\}$, and $\mathcal{M}_{n+1}$ be the set of all N which is a ground model or a generic extension of some $W \in \mathcal{M}_n$.
3. $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$ is the generic multiverse generated by M.

Remark

Let $\mathcal{M}$ be a Woodin's generic multiverse.

1. Since each $M \in \mathcal{M}$ is countable, $\mathcal{M}$ satisfies: For every $M \in \mathcal{M}$ and $\mathbb{P} \in M$, there is an $(M, \mathbb{P})$-generic G with $M[G] \in \mathcal{M}$.
2. For a given countable transitive $\in$-model M of ZFC, there is a unique Woodin's generic multiverse $\mathcal{M}$ with $M \in \mathcal{M}$. In this sense, $\mathcal{M}$ is the generic multiverse containing M, or the generic multiverse generated by M.

The construction of Woodin's GM

1. Fix a countable transitive $\in$-model M of ZFC.
2. Let $\mathcal{M}_0 = \{M\}$, and $\mathcal{M}_{n+1}$ be the set of all N which is a ground model or a generic extension of some $W \in \mathcal{M}_n$.
3. $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$ is the generic multiverse generated by M.

Steel's generic multiverse

Definition (Steel 2014)

A collection $\mathcal{M}$ is a Steel's generic multiverse if :

1. $\mathcal{M}$ is a family of transitive $\in$-models (not necessary countable, nor set) of ZFC.
2. $\mathcal{M}$ is closed under taking ground models.
3. For every $M \in \mathcal{M}$ and poset $\mathbb{P} \in \mathcal{M}$, there is an $(M, \mathbb{P})$-generic G with $M[G] \in \mathcal{M}$.
4. (Amalgamation) For every $M, N \in \mathcal{M}$ there is $W \in \mathcal{M}$ such that W is a common generic extension of M and N.

- Amalgamation represents "Every world has the same information".

Standard construction of Steel's generic multiverse

1. Fix a countable transitive $\in$-model M of ZFC.
2. Consider $\mathrm{Coll}(< ON)$ in M; it is a class forcing notion which adds a surjection from ω onto each ordinals in M.
3. Take an $(M, \mathrm{Coll}(< ON))$-generic G.
4. Let $\mathcal{M} = \{N \mid N \text{ is a ground model of } M[G_\alpha] \text{ for some } \alpha\}$, where $G_\alpha = G \cap \mathrm{Coll}(\alpha)$ is generic adding surjection from ω onto α.
5. $\mathcal{M}$ satisfies the conditions of Steel's generic multiverse.

Remark

Unlike Woodin's generic multiverse, this construction depends on the choice of an $(M, \mathrm{Coll}(< ON))$-generic G.

Standard construction of Steel's generic multiverse

1. Fix a countable transitive $\in$-model M of ZFC.
2. Consider $\mathrm{Coll}(< ON)$ in M; it is a class forcing notion which adds a surjection from ω onto each ordinals in M.
3. Take an $(M, \mathrm{Coll}(< ON))$-generic G.
4. Let $\mathcal{M} = \{N \mid N \text{ is a ground model of } M[G_\alpha] \text{ for some } \alpha\}$, where $G_\alpha = G \cap \mathrm{Coll}(\alpha)$ is generic adding surjection from ω onto α.
5. $\mathcal{M}$ satisfies the conditions of Steel's generic multiverse.

Remark

Unlike Woodin's generic multiverse, this construction depends on the choice of an $(M, \mathrm{Coll}(< ON))$-generic G.

- Both Woodin and Steel's generic multiverses need ZFC as a background theory.
- We want to develop the theory of generic multiverse without background ZFC.

Question

Can we develop a formal system, or axiomatization of Generic Multiverse? For instance, is there a first order (or some nice) theory T which axiomatizes (Woodin or Steel's) generic multiverse in the sense of Model theory?

- Väänänen developed multiverse logic using his dependence logic, and observed Steel's GM.
- Steel gave such a theory which characterize his GM.
- Steel's approach is more direct, so in this talk we will consider a variant of his approach.

- Both Woodin and Steel's generic multiverses need ZFC as a background theory.
- We want to develop the theory of generic multiverse without background ZFC.

Question

Can we develop a formal system, or axiomatization of Generic Multiverse? For instance, is there a first order (or some nice) theory T which axiomatizes (Woodin or Steel's) generic multiverse in the sense of Model theory?

- Väänänen developed multiverse logic using his dependence logic, and observed Steel's GM.
- Steel gave such a theory which characterize his GM.
- Steel's approach is more direct, so in this talk we will consider a variant of his approach.

- Both Woodin and Steel's generic multiverses need ZFC as a background theory.
- We want to develop the theory of generic multiverse without background ZFC.

Question

Can we develop a formal system, or axiomatization of Generic Multiverse? For instance, is there a first order (or some nice) theory T which axiomatizes (Woodin or Steel's) generic multiverse in the sense of Model theory?

- Väänänen developed multiverse logic using his dependence logic, and observed Steel's GM.
- Steel gave such a theory which characterize his GM.
- Steel's approach is more direct, so in this talk we will consider a variant of his approach.

Generic Multiverse logic $\mathbb{GM}$

- $\mathbb{GM}$ is a first order two-sorted logic:
  - set (first order) variables: $x, y, z, \dots$
  - world (class, second-order) variables: $M, N, W, \dots$.
  - predicate symbols: $\in$, $=$.
- Atomic formulas: $x = y$, $M = N$, $x \in y$, $x \in M$.
- A $\mathbb{GM}$-structure will be of the form $\mathcal{M} = (\mathcal{S}^{\mathcal{M}}, \mathcal{W}^{\mathcal{M}}, \in^{\mathcal{M}})$;
  the set-part $\mathcal{S}^{\mathcal{M}}$; a family of possible sets.
  the world-part $\mathcal{W} \subseteq \mathcal{P}(\mathcal{S}^{\mathcal{M}})$; a family of possible worlds.
- We define $\models$ and $\vdash$ by standard ways.
- A formula (sentence) of set-theory is a formula (sentence) of $\mathbb{GM}$ which does not contain world variables.

Axioms of Steel's generic multiverse

- T_S consists of:
  1. $\forall M(\sigma^M)$ where $\sigma \in \mathsf{ZFC}$.
  2. $\forall M \forall x \in M \forall y \in x(y \in M)$, $\forall x \exists M(x \in M)$
  3. $\forall M \forall N(M = N \leftrightarrow \forall x(x \in M \leftrightarrow x \in N))$.
  4. Closed under taking ground models;
     For every $M \in \mathcal{M}$ and ground N of M, we have $N \in \mathcal{M}$.
  5. Closed under taking generic extensions;
     For every $M \in \mathcal{M}$ and poset $\mathbb{P} \in M$, there is an $(M, \mathbb{P})$-generic G with $M[G] \in \mathcal{M}$.
  6. Amalgamation.

Definability of ground models

- How to describe "closed under taking ground models" in the language of $\mathbb{GM}$?

Fact (Laver, Woodin, Fuchs-Hamkins-Reitz, in ZFC or NBG)

There is a formula $\varphi_G(x, y)$ of set-theory such that:

1. For every $r \in V$, the definable class $W_r = \{x \mid \varphi_G(x, r)\}$ is a transitive model of ZFC and is a ground model of V.
2. For every transitive model $W \subseteq V$ of ZFC, if W is a ground model of V then $W = W_r$ for some $r \in W$.

So "closed under taking ground models" can be:

$\forall M \forall r \in M \exists N (N = \{x \in M \mid \varphi_G(x, r)^M\})$.

Definability of ground models

- How to describe "closed under taking ground models" in the language of $\mathbb{GM}$?

Fact (Laver, Woodin, Fuchs-Hamkins-Reitz, in ZFC or NBG)

There is a formula $\varphi_G(x, y)$ of set-theory such that:

1. For every $r \in V$, the definable class $W_r = \{x \mid \varphi_G(x, r)\}$ is a transitive model of ZFC and is a ground model of V.
2. For every transitive model $W \subseteq V$ of ZFC, if W is a ground model of V then $W = W_r$ for some $r \in W$.

So "closed under taking ground models" can be:

$$\forall M \forall r \in M \exists N (N = \{x \in M \mid \varphi_G(x, r)^M\}).$$

Axiom of Steel's GM T_S

1. $\forall M(\sigma^M)$ (where $\sigma \in$ ZFC).
2. $\forall M \forall x \in M \forall y \in x(y \in M)$, $\forall x \exists M(x \in M)$.
3. $\forall M \forall N(M = N \leftrightarrow \forall x(x \in M \leftrightarrow x \in N))$.
4. $\forall M \forall r \in M \exists N(N = \{x \in M \mid \varphi_G(x, r)^M\})$
5. $\forall M \forall \mathbb{P} \in M \exists g \exists N(g$ is $(M, \mathbb{P})$-generic $\wedge N = M[g])$.
6. $\forall M \forall N \exists W(M$ and N are ground models of $W)$.

Where

- g is $(M, \mathbb{P})$-generic $\iff g \subseteq \mathbb{P}$ is a filter $\wedge \forall D \in M(D$ is dense in $\mathbb{P} \to D \cap g \neq \emptyset)$.
- $N = M[g] \iff \mathbb{P}, g \in N \wedge \forall x \in N \exists \dot{\tau} \in M(\dot{\tau}$ is $\mathbb{P}$-name$\wedge \dot{\tau}_g = x)$.

It is easy to check that $\mathcal{M} \vDash T_S \iff$ the world part of $\mathcal{M}$ is a Steel's generic multiverse.

- Can Woodin's generic multiverse be axiomatized by a similar way?
- Let us forget the condition "countable transitive model".
- Countability may need to only guarantee that: every world has a generic extension.

The axiom T_W would be:

1. $\forall M(\sigma^M)$ (where $\sigma \in$ ZFC).
2. $\forall M \forall x \in M \forall y \in x(y \in M)$, $\forall x \exists M(x \in M)$.
3. $\forall M \forall N(M = N \leftrightarrow \forall x(x \in M \leftrightarrow x \in N))$.
4. $\forall M \forall r \in M \exists N(N = \{x \in M \mid \varphi_G(x, r)^M\})$
5. $\forall M \forall \mathbb{P} \in M \exists g \exists N(g$ is $(M, \mathbb{P})$-generic $\wedge N = M[g])$.
6. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in M$, and $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.
7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

- Can Woodin's generic multiverse be axiomatized by a similar way?
- Let us forget the condition "countable transitive model".
- Countability may need to only guarantee that: every world has a generic extension.

The axiom T_W would be:

1. $\forall M(\sigma^M)$ (where $\sigma \in \mathrm{ZFC}$).
2. $\forall M \forall x \in M \forall y \in x(y \in M)$, $\forall x \exists M(x \in M)$.
3. $\forall M \forall N(M = N \leftrightarrow \forall x(x \in M \leftrightarrow x \in N))$.
4. $\forall M \forall r \in M \exists N(N = \{x \in M \mid \varphi_G(x, r)^M\})$
5. $\forall M \forall \mathbb{P} \in M \exists g \exists N(g$ is $(M, \mathbb{P})$-generic $\wedge N = M[g])$.
6. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in M$, and $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.
7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

6. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in M$, $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.

This may be interpreted as:

Strong Closure

$\forall M \forall \mathbb{P} \in M \forall g (g \text{ is } (M, \mathbb{P})\text{-generic} \to \exists N (N = M[g]))$.

But this is not sufficient: generic filter g ranges only over the set-part of a model, but there might be a generic filter outside of a model.

7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

This is a serious problem: "finite paths" can not be described by our logic!

6. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in M$, $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.

This may be interpreted as:

Strong Closure

$\forall M \forall \mathbb{P} \in M \forall g (g \text{ is } (M, \mathbb{P})\text{-generic} \to \exists N (N = M[g]))$.

But this is not sufficient: generic filter g ranges only over the set-part of a model, but there might be a generic filter outside of a model.

7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

This is a serious problem: "finite paths" can not be described by our logic!

6. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in M$, $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.

This may be interpreted as:

Strong Closure

$\forall M \forall \mathbb{P} \in M \forall g (g \text{ is } (M, \mathbb{P})\text{-generic} \rightarrow \exists N (N = M[g]))$.

But this is not sufficient: generic filter g ranges only over the set-part of a model, but there might be a generic filter outside of a model.

7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

This is a serious problem: "finite paths" can not be described by our logic!

6. For every $M \in \mathcal{M}$, poset $\mathbb{P} \in M$, $(M, \mathbb{P})$-generic G, we have $M[G] \in \mathcal{M}$.

This may be interpreted as:

Strong Closure

$\forall M \forall \mathbb{P} \in M \forall g(g \text{ is } (M, \mathbb{P})\text{-generic} \rightarrow \exists N(N = M[g]))$.

But this is not sufficient: generic filter g ranges only over the set-part of a model, but there might be a generic filter outside of a model.

7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

This is a serious problem: "finite paths" can not be described by our logic!

The downward directedness of grounds

- The downward directedness of grounds helps this problem.

Theorem (Usuba, downward directedness of grounds, in ZFC)

For every models M and N of ZFC, if M and N are ground models of some supermodel, then M and N has a common ground model W of M and N.

Corollary

If $\mathcal{M}$ is a Woodin or Steel's generic multiverse, then for every $M, N \in \mathcal{M}$ there is W which is a common ground model of M and N.

- If $\mathcal{M}$ has the amalgamation property, it is immediate.
- If $\mathcal{M}$ is a Woodin's GM, then it follows from the construction.

The downward directedness of grounds

- The downward directedness of grounds helps this problem.

Theorem (Usuba, downward directedness of grounds, in ZFC)

For every models M and N of ZFC, if M and N are ground models of some supermodel, then M and N has a common ground model W of M and N.

Corollary

If $\mathcal{M}$ is a Woodin or Steel's generic multiverse, then for every $M, N \in \mathcal{M}$ there is W which is a common ground model of M and N.

- If $\mathcal{M}$ has the amalgamation property, it is immediate.
- If $\mathcal{M}$ is a Woodin's GM, then it follows from the construction.

The downward directedness of grounds

- The downward directedness of grounds helps this problem.

Theorem (Usuba, downward directedness of grounds, in ZFC)

For every models M and N of ZFC, if M and N are ground models of some supermodel, then M and N has a common ground model W of M and N.

Corollary

If $\mathcal{M}$ is a Woodin or Steel's generic multiverse, then for every $M, N \in \mathcal{M}$ there is W which is a common ground model of M and N.

- If $\mathcal{M}$ has the amalgamation property, it is immediate.
- If $\mathcal{M}$ is a Woodin's GM, then it follows from the construction.

The downward directedness of grounds

- The downward directedness of grounds helps this problem.

Theorem (Usuba, downward directedness of grounds, in ZFC)

For every models M and N of ZFC, if M and N are ground models of some supermodel, then M and N has a common ground model W of M and N.

Corollary

If $\mathcal{M}$ is a Woodin or Steel's generic multiverse, then for every $M, N \in \mathcal{M}$ there is W which is a common ground model of M and N.

- If $\mathcal{M}$ has the amalgamation property, it is immediate.
- If $\mathcal{M}$ is a Woodin's GM, then it follows from the construction.

7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

is equivalent to

Every $M, N \in \mathcal{M}$ have a common ground model.

This can be described by:

$\forall M \forall N \exists W (W$ is a ground model of M and $N)$.

7. Each $M \in \mathcal{M}$ is connected with other $N \in \mathcal{M}$ by finite paths.

is equivalent to

Every $M, N \in \mathcal{M}$ have a common ground model.

This can be described by:

$\forall M \forall N \exists W$($W$ is a ground model of M and N).

Axioms of Woodin's GM T_W (approximation)

1. $\forall M(\sigma^M)$ (where $\sigma \in \mathrm{ZFC}$).
2. $\forall M \forall x \in M \forall y \in x(y \in M)$, $\forall x \exists M(x \in M)$.
3. $\forall M \forall N(M = N \leftrightarrow \forall x(x \in M \leftrightarrow x \in N))$.
4. $\forall M \forall r \in M \exists N(N = \{x \in M \mid \varphi_G(x, r)^M\})$
5. $\forall M \forall \mathbb{P} \in M \exists g \exists N(g \text{ is } (M, \mathbb{P})\text{-generic} \wedge N = M[g])$.
6. $\forall M \forall \mathbb{P} \in M \forall g(g \text{ is } (M, \mathbb{P})\text{-generic} \rightarrow \exists N(N = M[g]))$.
7. $\forall M \forall N \exists W(W \text{ is a ground model of } M \text{ and } N)$.

- Woodin's generic multiverse is a model of T_W.
- However the converse does not hold.

The basic GM

Let us consider the intersection of T_S and T_W, which grabs the essence of Generic Multiverse.

Axioms of basic GM, T_{GM}

1. $\forall M(\sigma^M)$ (where $\sigma \in \mathrm{ZFC}$).
2. $\forall M \forall x \in M \forall y \in x(y \in M)$, $\forall x \exists M(x \in M)$.
3. $\forall M \forall N(M = N \leftrightarrow \forall x(x \in M \leftrightarrow x \in N))$.
4. $\forall M \forall r \in M \exists N(N = \{x \in M \mid \varphi_G(x, r)^M\})$
5. $\forall M \forall \mathbb{P} \exists g \exists N(g \text{ is } (M, \mathbb{P})\text{-generic} \wedge N = M[g])$.
6. $\forall M \forall N \exists W(W \text{ is a ground model of } M \text{ and } N)$.

- $T_W = T_{GM}$+Strong Closure: $\forall M \forall \mathbb{P} \forall g(g \text{ is } (M, \mathbb{P})\text{-generic} \rightarrow \exists N(N = M[g]))$.
- $T_S = T_{GM}$+Amalgamation.

Convention

For a model $\mathcal{M}$ of T_{GM}, $x \in \mathcal{M}$ means that x is an element of the set-part of $\mathcal{M}$, and $M \in \mathcal{M}$ means that M is of the world-part. For $x, y, M \in \mathcal{M}$, $x \in y$ means $\mathcal{M} \vDash x \in y$, and $x \in M$ does $\mathcal{M} \vDash x \in M$.

Lemma (Forcing Theorem)

For every model $\mathcal{M}$ of T_{GM}, $M \in \mathcal{M}$, $\mathbb{P} \in M$, and sentence σ of set-theory, the following are equivalent:

1. $\mathcal{M} \vDash (\Vdash_{\mathbb{P}} \sigma)^M$.
2. $\mathcal{M} \vDash \forall N \forall g (g \text{ is } (M, \mathbb{P})\text{-generic} \wedge N = M[g] \rightarrow \sigma^N)$.

Convention

For a model $\mathcal{M}$ of T_{GM}, $x \in \mathcal{M}$ means that x is an element of the set-part of $\mathcal{M}$, and $M \in \mathcal{M}$ means that M is of the world-part. For $x, y, M \in \mathcal{M}$, $x \in y$ means $\mathcal{M} \vDash x \in y$, and $x \in M$ does $\mathcal{M} \vDash x \in M$.

Lemma (Forcing Theorem)

For every model $\mathcal{M}$ of T_{GM}, $M \in \mathcal{M}$, $\mathbb{P} \in M$, and sentence σ of set-theory, the following are equivalent:

1. $\mathcal{M} \vDash (\Vdash_{\mathbb{P}} \sigma)^M$.
2. $\mathcal{M} \vDash \forall N \forall g (g \text{ is } (M, \mathbb{P})\text{-generic} \wedge N = M[g] \to \sigma^N)$.

Definition (Woodin)

A sentence σ of set-theory is a multiverse truth if σ is true in any world of a generic multiverse.

Fact (Woodin)

There is a computable translation $\sigma \to \sigma^*$ such that for every Woodin's GM $\mathcal{M}$, σ is a multiverse truth (in $\mathcal{M}$)

$\iff$ $M \vDash \sigma^*$ for some $M \in \mathcal{M}$

$\iff$ $M \vDash \sigma^*$ for every $M \in \mathcal{M}$.

Lemma

There is a computable translation $\sigma \to \sigma^*$ such that for every model $\mathcal{M}$ of T_{GS}, σ is a multiverse truth (in $\mathcal{M}$)

$\iff$ $\mathcal{M} \vDash \exists M((\sigma^*)^M)$.

$\iff$ $\mathcal{M} \vDash \forall M((\sigma^*)^M)$.

Definition (Woodin)

A sentence σ of set-theory is a multiverse truth if σ is true in any world of a generic multiverse.

Fact (Woodin)

There is a computable translation $\sigma \to \sigma^*$ such that for every Woodin's GM $\mathcal{M}$, σ is a multiverse truth (in $\mathcal{M}$)
$\iff$ $M \models \sigma^*$ for some $M \in \mathcal{M}$
$\iff$ $M \models \sigma^*$ for every $M \in \mathcal{M}$.

Lemma

There is a computable translation $\sigma \to \sigma^*$ such that for every model $\mathcal{M}$ of T_{GS}, σ is a multiverse truth (in $\mathcal{M}$)
$\iff$ $\mathcal{M} \models \exists M((\sigma^*)^M)$.
$\iff$ $\mathcal{M} \models \forall M((\sigma^*)^M)$.

Definition (Woodin)

A sentence σ of set-theory is a multiverse truth if σ is true in any world of a generic multiverse.

Fact (Woodin)

There is a computable translation $\sigma \to \sigma^*$ such that for every Woodin's GM $\mathcal{M}$, σ is a multiverse truth (in $\mathcal{M}$)

$\iff$ $M \models \sigma^*$ for some $M \in \mathcal{M}$

$\iff$ $M \models \sigma^*$ for every $M \in \mathcal{M}$.

Lemma

There is a computable translation $\sigma \to \sigma^*$ such that for every model $\mathcal{M}$ of T_{GS}, σ is a multiverse truth (in $\mathcal{M}$)

$\iff \mathcal{M} \models \exists M((\sigma^*)^M)$.

$\iff \mathcal{M} \models \forall M((\sigma^*)^M)$.

General question

Let $\mathcal{M}$ be a model of T_{GM}. What is the structure of the set-part of $\mathcal{M}$?

Fact (Mostowski, Woodin)

Let M be a countable transitive $\in$-model of ZFC. Then M has two generic extensions N_0, N_1 such that there is no common generic extension of N_0 and N_1.

Corollary

Let $\mathcal{M}$ be a Woodin's generic multiverse.

1. $\mathcal{M}$ cannot satisfy Amalgamation.
2. Indeed the set-part of $\mathcal{M}$ does not satisfy the paring axiom $\forall x \forall y \exists z (z = \{x, y\})$. So $T_{GM} + \neg$Paring is consistent.

- Under T_{GM}, Paring $\iff \forall x \forall y \exists M (x, y \in M)$.

General question

Let $\mathcal{M}$ be a model of T_{GM}. What is the structure of the set-part of $\mathcal{M}$?

Fact (Mostowski, Woodin)

Let M be a countable transitive $\in$-model of ZFC. Then M has two generic extensions N_0, N_1 such that there is no common generic extension of N_0 and N_1.

Corollary

Let $\mathcal{M}$ be a Woodin's generic multiverse.

1. $\mathcal{M}$ cannot satisfy Amalgamation.
2. Indeed the set-part of $\mathcal{M}$ does not satisfy the paring axiom $\forall x \forall y \exists z (z = \{x, y\})$. So $T_{GM} + \neg$Paring is consistent.

- Under T_{GM}, Paring $\iff \forall x \forall y \exists M (x, y \in M)$.

General question

Let $\mathcal{M}$ be a model of T_{GM}. What is the structure of the set-part of $\mathcal{M}$?

Fact (Mostowski, Woodin)

Let M be a countable transitive $\in$-model of ZFC. Then M has two generic extensions N_0, N_1 such that there is no common generic extension of N_0 and N_1.

Corollary

Let $\mathcal{M}$ be a Woodin's generic multiverse.

1. $\mathcal{M}$ cannot satisfy Amalgamation.
2. Indeed the set-part of $\mathcal{M}$ does not satisfy the paring axiom $\forall x \forall y \exists z (z = \{x, y\})$. So $T_{GM} + \neg$Paring is consistent.

- Under T_{GM}, Paring $\iff \forall x \forall y \exists M (x, y \in M)$.

In the standard construction of Steel's GM, the set-part is a class forcing extension of some world via $\mathrm{Coll}(< ON)$.

Fact

The class forcing $\mathrm{Coll}(< ON)$ forces ZFC−Power set Axiom+"every set is countable".

Lemma

Let $\mathcal{M}$ be a Steel's generic multiverse by the standard construction. Then the set-part of $\mathcal{M}$ is a model of ZFC−Power set Axiom+"every set is countable".

Question

What is the structure of the set-part of a model of T_{GM}+Amalgamation?

In the standard construction of Steel's GM, the set-part is a class forcing extension of some world via $\mathrm{Coll}(< ON)$.

Fact

The class forcing $\mathrm{Coll}(< ON)$ forces ZFC$-$Power set Axiom$+$"every set is countable".

Lemma

Let $\mathcal{M}$ be a Steel's generic multiverse by the standard construction. Then the set-part of $\mathcal{M}$ is a model of ZFC$-$Power set Axiom$+$"every set is countable".

Question

What is the structure of the set-part of a model of T_{GM}+Amalgamation?

In the standard construction of Steel's GM, the set-part is a class forcing extension of some world via $\mathrm{Coll}(< ON)$.

Fact

The class forcing $\mathrm{Coll}(< ON)$ forces ZFC$-$Power set Axiom$+$"every set is countable".

Lemma

Let $\mathcal{M}$ be a Steel's generic multiverse by the standard construction. Then the set-part of $\mathcal{M}$ is a model of ZFC$-$*Power set Axiom*$+$*"every set is countable".*

Question

What is the structure of the set-part of a model of T_{GM}+Amalgamation?

In the standard construction of Steel's GM, the set-part is a class forcing extension of some world via $\mathrm{Coll}(< ON)$.

Fact

The class forcing $\mathrm{Coll}(< ON)$ forces ZFC$-$Power set Axiom$+$"every set is countable".

Lemma

Let $\mathcal{M}$ be a Steel's generic multiverse by the standard construction. Then the set-part of $\mathcal{M}$ is a model of ZFC$-$Power set Axiom$+$"every set is countable".

Question

What is the structure of the set-part of a model of T_{GM}+Amalgamation?

Theorem

Let $\mathcal{M}$ be a model of T_{GM}. If the set-part of $\mathcal{M}$ satisfies Paring, then for every $x_0, \ldots, x_n \in \mathcal{M}$ and formula φ of set-theory, the following are equivalent:

1. The set-part of $\mathcal{M}$ satisfies $\varphi(x_0, \ldots, x_n)$.
2. For every $M \in \mathcal{M}$,
   $\mathcal{M} \models x_0, \ldots, x_n \in M \rightarrow (\Vdash_{\mathrm{Coll}(<ON)} \varphi(x_0, \ldots, x_n))^M$.
3. There exists $M \in \mathcal{M}$ such that
   $\mathcal{M} \models x_0, \ldots, x_n \in M \wedge (\Vdash_{\mathrm{Coll}(<ON)} \varphi(x_0, \ldots, x_n))^M$.

Roughly speaking, if $\mathcal{M}$ satisfies Paring, then each world knows the truths of the set-part.

Corollary

Let $\mathcal{M}$ be a model of T_{GM}.

1. If the set-part of $\mathcal{M}$ satisfies Paring, then it is a model of ZFC−Power set Axiom+"every set is countable":

   T_{GM} + Paring $\vdash$ ZFC−Power set+"every set is countable".

2. In particular T_{GM}+Amalgamation$\vdash$ ZFC−Power set+"every set is countable".

Sketch of the proof

By induction on the complexity of the formula φ.

- If φ is atomic, it is clear.
- The boolean combination step is easy.
- Suppose $\varphi = \exists x \psi(x)$.
  - If $\mathrm{Coll}(< ON)$ forces φ over M, then there is α and a generic extension $M[g] \in \mathcal{M}$ of M via $\mathrm{Coll}(\alpha)$, and $y \in M[g]$ such that $\mathrm{Coll}(< ON)$ forces $\psi(y)$ over $M[g]$. Then $\psi(y)$ holds in the set-part of $\mathcal{M}$.
  - If φ holds in the set-part of $\mathcal{M}$, pick a witness $z \in \mathcal{M}$.
  - Choose $N \in \mathcal{M}$ with $z \in N$, and $W \in \mathcal{M}$ which is a common ground model of M and N.
  - Then $\mathrm{Coll}(< ON)$ forces φ over W, and so does over M.

- Amalgamation is a Π^1_2-statement.
- Paring is a Π^0_2-statement, so the complexity is drastically reduced.

Question

Is T_{GM}+Paring equivalent to T_S?

T_{GM}+Paring$\vdash$Amalgamation?

Theorem

It is consistent that T_{GM}+Paring+$\neg$Amalgamation.

- Amalgamation is a Π^1_2-statement.
- Paring is a Π^0_2-statement, so the complexity is drastically reduced.

Question

Is T_{GM}+Paring equivalent to T_S?

T_{GM}+Paring$\vdash$Amalgamation?

Theorem

It is consistent that T_{GM}+Paring+$\neg$Amalgamation.

- Amalgamation is a Π^1_2-statement.
- Paring is a Π^0_2-statement, so the complexity is drastically reduced.

Question

Is T_{GM}+Paring equivalent to T_S?

T_{GM}+Paring$\vdash$Amalgamation?

Theorem

It is consistent that T_{GM}+Paring+$\neg$Amalgamation.

Minimum ground

Fact (Fuchs-Hamkins-Reitz)

1. It is consistent that ZFC+"there exists the minimum ground model".
2. It is consistent that ZFC+Large Cardinal Axiom+"there exists the minimum ground model".
3. It is consistent that ZFC+"there is no minimal ground models".

Remark

By the definability of all ground models, the statement "there exists the minimum ground model" is a sentence of set-theory.

Theorem

Let $\mathcal{M}$ be a model of T_{GS}. Then the following are equivalent:

1. $\mathcal{M}$ has the minimum world.
2. There is $M \in \mathcal{M}$ such that "there exists the minimum ground model" holds in M.
3. For every $M \in \mathcal{M}$, "there exists the minimum ground model" holds in M.

Actually, for $M \in \mathcal{M}$, if the intersection of all ground models of M is a ground model of M, then it is the minimum world of $\mathcal{M}$.

In particular, there is a sentence of set-theory σ such that

$\mathcal{M}$ has the minimum world

$\iff \mathcal{M} \models \exists N \forall M (N \subseteq M) \iff \mathcal{M} \models \exists M (\sigma^M) \iff \mathcal{M} \models \forall M (\sigma^M)$.

So each world knows whether there is the minimum world or not.

Proposition

T_{GM}+Paring+"there exists the minimum world" $\vdash$Amalgamation.

Hence under T_{GM}+"there exists the minimum world", Pairing is equivalent to Amalgamation.

Proof.

1. Let $W_0 \in \mathcal{M}$ be the minimum world.
2. Take $M, N \in \mathcal{M}$. Then $M = W_0[g]$ and $N = W_0[h]$ for some generic $g, h \in \mathcal{M}$.
3. By Pairing, there is $W \in \mathcal{M}$ with $g, h \in W$.
4. Then $W_0 \subseteq W$ and $g, h \in W$, so $M, N \subseteq W$.

□

Large cardinal

Theorem (Usuba, in ZFC)

If there exists an extendible cardinal, then V has the minimum ground model.

An uncountable cardinal κ is extendible if for every $\alpha \geq \kappa$, there are β and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $\alpha < j(\kappa)$.

Corollary

T_{GM}+Paring+"some world has a large cardinal" $\vdash$Amalgamation.

Large cardinal

Theorem (Usuba, in ZFC)

If there exists an extendible cardinal, then V has the minimum ground model.

An uncountable cardinal κ is extendible if for every $\alpha \geq \kappa$, there are β and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $\alpha < j(\kappa)$.

Corollary

T_{GM}+Paring+"some world has a large cardinal" $\vdash$Amalgamation.

Sketch of a construction of T_{GM}+Paring+$\neg$Amalgamation

1. Fix a countable transitive $\in$-model M of ZFC.
2. Take a class Easton forcing extension $M[H]$ of M.
3. It is known that $M[H]$ has no minimal ground models.
4. Do the standard construction of Steel's GM starting from $M[H]$.
5. Let $\mathcal{M}$ be a Steel's GM.
6. Remove some worlds from $\mathcal{M}$ carefully.
7. By careful removing, we can get a model $\mathcal{N}$ of T_{GM}+Paring+$\neg$Amalgamation.

- Our example $\mathcal{N}$ is a sub-multiverse of some model of T_S.

Definition

Let $\mathcal{M} = (\mathcal{S}, \mathcal{W})$ and $\mathcal{M}' = (\mathcal{S}', \mathcal{W}')$ be models of T_{GM}. $\mathcal{M}'$ is a worlds-extension of $\mathcal{M}$ if $\mathcal{S} = \mathcal{S}'$ and $\mathcal{W} \subseteq \mathcal{W}'$.

Question (Open)

Does every model of T_{GM}+Paring have a worlds-extension which is a model of T_{GM}+Amalgamation?

Remark

T_{GM}+Paring+$\neg$Strong Closure is consistent; $\mathcal{N}$ above is a witness.

Proposition

Let $\mathcal{M}$ be a model of T_{GS}. If $\mathcal{M}$ satisfies Paring, then $\mathcal{M}$ has a worlds-extension satisfying Strong Closure+Paring.

- Our example $\mathcal{N}$ is a sub-multiverse of some model of T_S.

Definition

Let $\mathcal{M} = (\mathcal{S}, \mathcal{W})$ and $\mathcal{M}' = (\mathcal{S}', \mathcal{W}')$ be models of T_{GM}. $\mathcal{M}'$ is a worlds-extension of $\mathcal{M}$ if $\mathcal{S} = \mathcal{S}'$ and $\mathcal{W} \subseteq \mathcal{W}'$.

Question (Open)

Does every model of T_{GM}+Paring have a worlds-extension which is a model of T_{GM}+Amalgamation?

Remark

T_{GM}+Paring+$\neg$Strong Closure is consistent; $\mathcal{N}$ above is a witness.

Proposition

Let $\mathcal{M}$ be a model of T_{GS}. If $\mathcal{M}$ satisfies Paring, then $\mathcal{M}$ has a worlds-extension satisfying Strong Closure+Paring.

- Our example $\mathcal{N}$ is a sub-multiverse of some model of T_S.

Definition

Let $\mathcal{M} = (\mathcal{S}, \mathcal{W})$ and $\mathcal{M}' = (\mathcal{S}', \mathcal{W}')$ be models of T_{GM}. $\mathcal{M}'$ is a worlds-extension of $\mathcal{M}$ if $\mathcal{S} = \mathcal{S}'$ and $\mathcal{W} \subseteq \mathcal{W}'$.

Question (Open)

Does every model of T_{GM}+Paring have a worlds-extension which is a model of T_{GM}+Amalgamation?

Remark

T_{GM}+Paring+$\neg$Strong Closure is consistent; $\mathcal{N}$ above is a witness.

Proposition

Let $\mathcal{M}$ be a model of T_{GS}. If $\mathcal{M}$ satisfies Paring, then $\mathcal{M}$ has a worlds-extension satisfying Strong Closure+Paring.

- Our example $\mathcal{N}$ is a sub-multiverse of some model of T_S.

Definition

Let $\mathcal{M} = (\mathcal{S}, \mathcal{W})$ and $\mathcal{M}' = (\mathcal{S}', \mathcal{W}')$ be models of T_{GM}. $\mathcal{M}'$ is a worlds-extension of $\mathcal{M}$ if $\mathcal{S} = \mathcal{S}'$ and $\mathcal{W} \subseteq \mathcal{W}'$.

Question (Open)

Does every model of T_{GM}+Paring have a worlds-extension which is a model of T_{GM}+Amalgamation?

Remark

T_{GM}+Paring+$\neg$Strong Closure is consistent; $\mathcal{N}$ above is a witness.

Proposition

Let $\mathcal{M}$ be a model of T_{GS}. If $\mathcal{M}$ satisfies Paring, then $\mathcal{M}$ has a worlds-extension satisfying Strong Closure+Paring.

1. Note that since $\mathcal{M}$ is a model of T_{GS}+Paring, the set-part of $\mathcal{M}$ is a model of ZFC$-$Power set.
2. Hence for every $M \in \mathcal{M}$, $\mathbb{P} \in M$, and $(M, \mathbb{P})$-generic $g \in \mathcal{M}$, $M[g]$ is (second-order) definable class in $\mathcal{M}$, and by forcing theorem, $M[g]$ is a model of ZFC.
3. Just let $\mathcal{W}' = \{M[g] \mid M \in \mathcal{M}, g \in \mathcal{M}$ is $(M, \mathbb{P})$-generic for some $\mathbb{P} \in M\}$, and $\mathcal{M}' = (\mathcal{S}, \mathcal{W}')$.
4. Point is: for every $M \in \mathcal{M}$, $\alpha \in M$, $(M, \mathrm{Coll}(\alpha))$-generic g, and $\beta \in M$, there is $g' \in \mathcal{M}$ which is $(M, \mathrm{coll}(\beta))$-generic and $M[g]$ is a ground model of $M[g']$.
5. This guarantees that $\mathcal{M}' = (\mathcal{S}, \mathcal{W}')$ is a model of T_{GS}+Paring+Strong Closure.

Remark

1. For a model $\mathcal{M}$ be a model of T_{GM}+Paring, let $\mathcal{M}' = (\mathcal{S}, \mathcal{W}')$ be a world extension of $\mathcal{M}$ constructed as above. Then $\mathcal{W}'$ is (second-order) definable in $\mathcal{M}$.
2. In this sense, the worlds-extension $\mathcal{M}'$ is a very weak extension of $\mathcal{M}$; There is a computable translation $\sigma \mapsto \sigma^*$ for sentences σ in $\mathbb{GM}$ such that $\mathcal{M}' \vDash \sigma \iff \mathcal{M} \vDash \sigma^*$. So the truth of $\mathcal{M}'$ is definable in $\mathcal{M}$.
3. In particular
   T_{GM}+Paring+Strong Closure$\vdash \sigma \iff T_{GM}$+Paring$\vdash \sigma^*$.

If we strengthen Paring to Amalgamation, we have:

Lemma

T_{GM}+Amalgamation$\vdash$Strong Closure.

In particular, every model of T_S is a model of T_W.

Remark

1. For a model $\mathcal{M}$ be a model of T_{GM}+Paring, let $\mathcal{M}' = (\mathcal{S}, \mathcal{W}')$ be a world extension of $\mathcal{M}$ constructed as above. Then $\mathcal{W}'$ is (second-order) definable in $\mathcal{M}$.
2. In this sense, the worlds-extension $\mathcal{M}'$ is a very weak extension of $\mathcal{M}$; There is a computable translation $\sigma \mapsto \sigma^*$ for sentences σ in $\mathbb{GM}$ such that $\mathcal{M}' \vDash \sigma \iff \mathcal{M} \vDash \sigma^*$. So the truth of $\mathcal{M}'$ is definable in $\mathcal{M}$.
3. In particular
   T_{GM}+Paring+Strong Closure$\vdash \sigma \iff T_{GM}$+Paring$\vdash \sigma^*$.

If we strengthen Paring to Amalgamation, we have:

Lemma

T_{GM}+Amalgamation$\vdash$Strong Closure.

In particular, every model of T_S is a model of T_W.

Definability of worlds

- In ZFC, every ground model is definable.
- In T_{GM}+Paring, the set-part is close to a class forcing extension of each world.

Question

Let $\mathcal{M}$ be a model of T_{GM}+Paring. Is each world definable in the set-part by a formula of set-theory? That is, is there a formula φ of set-theory such that for every $M \in \mathcal{M}$,

$$\mathcal{M} \models M = \{x \in S^{\mathcal{M}} \mid \varphi(x, r)\}$$

for some $r \in \mathcal{M}$?

- It is true if $\mathcal{M}$ has a world of $V = L$.
- If this is possible, then the world-part is definable in the set-part, so it become a redundant part...

Definability of worlds

- In ZFC, every ground model is definable.
- In T_{GM}+Paring, the set-part is close to a class forcing extension of each world.

Question

Let $\mathcal{M}$ be a model of T_{GM}+Paring. Is each world definable in the set-part by a formula of set-theory? That is, is there a formula φ of set-theory such that for every $M \in \mathcal{M}$,

$$\mathcal{M} \models M = \{x \in S^{\mathcal{M}} \mid \varphi(x, r)\}$$

for some $r \in \mathcal{M}$?

- It is true if $\mathcal{M}$ has a world of $V = L$.
- If this is possible, then the world-part is definable in the set-part, so it become a redundant part...

Definability of worlds

- In ZFC, every ground model is definable.
- In T_{GM}+Paring, the set-part is close to a class forcing extension of each world.

Question

Let $\mathcal{M}$ be a model of T_{GM}+Paring. Is each world definable in the set-part by a formula of set-theory? That is, is there a formula φ of set-theory such that for every $M \in \mathcal{M}$,

$$\mathcal{M} \models M = \{x \in S^{\mathcal{M}} \mid \varphi(x, r)\}$$

for some $r \in \mathcal{M}$?

- It is true if $\mathcal{M}$ has a world of $V = L$.
- If this is possible, then the world-part is definable in the set-part, so it become a redundant part...

Theorem

There is a model $\mathcal{M}$ of T_{GM}+Amalgamation and $M \in \mathcal{M}$ such that for every formula φ of set-theory and $r \in \mathcal{M}$, we have

$$\mathcal{M} \models M \neq \{x \in S^{\mathcal{M}} \mid \varphi(x, r)\}.$$

Hence M is never definable in $\mathcal{M}$ by a formula of set-theory.

- However our model $\mathcal{M}$ above has no minimal worlds.

Question (Open)

Suppose $\mathcal{M}$ is a model of T_{GM}+Amalgamation. If $\mathcal{M}$ has the minimum world, then is the minimum world definable in $\mathcal{M}$ by a formula of set-theory?

Theorem

There is a model $\mathcal{M}$ of T_{GM}+Amalgamation and $M \in \mathcal{M}$ such that for every formula φ of set-theory and $r \in \mathcal{M}$, we have

$$\mathcal{M} \models M \neq \{x \in S^{\mathcal{M}} \mid \varphi(x, r)\}.$$

Hence M is never definable in $\mathcal{M}$ by a formula of set-theory.

- However our model $\mathcal{M}$ above has no minimal worlds.

Question (Open)

Suppose $\mathcal{M}$ is a model of T_{GM}+Amalgamation. If $\mathcal{M}$ has the minimum world, then is the minimum world definable in $\mathcal{M}$ by a formula of set-theory?

Other questions

- Can Woodin's GM be axiomatized exactly by some formal way?
- In T_{GS}+Paring, the set-part is a model of ZFC−Power set, a first order set-theory. Does T_{GS}+Paring imply the replacement scheme for the formulas of $\mathbb{GM}$? Namely, for every formula φ of $\mathbb{GM}$, does T_{GS}+Paring imply $\forall \vec{M} \forall \vec{x} \forall y (\forall z \in y \exists ! w \varphi(z, w, \vec{x}, \vec{M}) \to \exists v \forall z \in y \exists z \in v \varphi(z, w, \vec{x}, \vec{M}))$?
- It is known that T_{GM}+Amalgamation implies it.

Conslusions

- Generic Multiverse Logic $\mathbb{GM}$.
- Basic Theory of Generic Multiverse T_{GM}, this grabs the essence of Generic Multiverse.
- Paring plays important roles in this context.
- The existence of the minimum world (or Large Cardinal Axiom) simplify the structure of GM.
- However the deference between T_{GM}+Paring and T_S is not so clear now.

References

- G. Fuchs, J. D. Hamkins, J. Reitz, Set-theoretic geology. Ann. Pure Appl. Logic 166 (2015), no. 4, 464–501.
- J. Steel, Gödel's program. in: Interpreting Gödel Critical Essays, Cambridge University Press, 2014.
- T. Usuba, The downward directed grounds hypothesis and very large cardinals. J. Math. Logic 17, 1750009 (2017)
- T. Usuba, Extendible cardinals and the mantle. To appear in Arch. Math. Logic.
- J. Väänänen, Multiverse set theory and absolutely undecidable propositions. in: Interpreting Gödel Critical Essays, Cambridge University Press, 2014.
- W. H. Woodin, The continuum hypothesis, the generic-multiverse of sets, and the Ω-conjecture. Set theory, arithmetic, and foundations of mathematics: theorems, philosophies, 1–42, Lect. Notes Log., 36, Assoc. Symbol. Logic, La Jolla, CA, 2011.

Thank you for your attention!