How to prove it: Phase transitions in logic

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SLS2016, Sendai, 29 January 2016

Overview:

1. Introduction
2. Lower bounds
3. Upper bounds
4. Sharpening

Introduction

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Phase transitions for incompleteness are results of the following form:

1. $T \nvdash \varphi_{f_n}$, but
2. $T \vdash \varphi_f$,

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In this talk we will treat the heuristics for these results. We will use the easy example of miniaturised Dickson's lemma to illustrate.

Introduction

Definition

Given $a, b \in \mathbb{N}^c$:

1. $|.|$ denotes the sup-norm: $|a| = \max_{i<c}(a)_i$,
2. $\leq$ denotes coordinatewise ordering:

$$a \leq b \Leftrightarrow (a)_0 \leq (b)_0 \wedge \cdots \wedge (a)_{c-1} \leq (b)_{c-1}.$$

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Definition (MDL_f)

For every l, c there exists $D =: D_f(c, l)$ such that for every sequence $m_0, \ldots, m_D$ of c-tuples, with $|m_i| \leq l + f(i)$, there exist $i < j \leq D$ with $m_i \leq m_j$.

Introduction

Fact

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Define $A_0(x) = x + 1$ *and* $A_{n+1}(x) = A_n^{(x)}(x)$*. For every primitive recursive function* f *there exists* n *such that:*

$$f \leq A_n$$

Definition

We call a sequence $m_0, \dots, m_R$ bad if $m_i \not\leq m_j$ for all $i < j \leq R$.

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Corollary

$\mathrm{RCA}_0 \nvdash \mathrm{MDL}_{\mathrm{id}}$.

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Our goal is to classify some $f\colon \mathbb{N} \to \mathbb{N}$ according to the provability of MDL_f.

Introduction: heuristics

Examine the following function:

$$a \mapsto D_{x \mapsto a}(c, l).$$

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Exercise

$D_{x \mapsto a}(c, l) = (l + a + 1)^c.$

Introduction: heuristics

Observe: Take $I_n(a) = a^n$, for every n there exist c, I such that:

$$I_n \leq a \mapsto D_{x \mapsto a}(c, I).$$

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Observe: Take $l_n(a) = a^n$, for every n there exist c, l such that:

$$l_n \leq a \mapsto D_{x \mapsto a}(c, l).$$

Take $u(a) = 2^a$, for every c, l:

$$a \mapsto D_{x \mapsto a}(c, l) \leq u.$$

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Experience suggests that using l_n^{-1} as a parameter value will result in *independence*, whilst using u^{-1} will result in *provability*.

Introduction: heuristics

Theorem

1. $\mathrm{RCA}_0 \nvdash \mathrm{MDL}_{\sqrt[n]{}}$, *but*
2. $\mathrm{RCA}_0 \vdash \mathrm{MDL}_{\log}$.

Lower bounds

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Exercise

For $l >?$:

$$D_{\mathrm{id}}(c, l) \leq D_{\sqrt[n]{}}(c + n + 1, l).$$

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Corollary

$\mathrm{RCA}_0 \nvdash \mathrm{MDL}_{\sqrt[n]{}}$.

Upper bounds

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If $i \leq 2^l$*, then* $\log i \leq l$*, so for* $l >?$*:*

$$D_{\log}(c, l) \leq D_l(c, l) \leq 2^l.$$

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Sharpening

Sharpening:lower bounds

Lemma

$\mathrm{RCA}_0 \nvdash \mathrm{MDL}_f$ *for* $f(x) = \sqrt[A^{-1}(x)]{x}$.

Proof: We examine D_f. Assume, for a contradiction, that:

$$D_f(2l+2, l+1) \leq A(l),$$

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By earlier proofs, we already know:

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Contradiction! □

Sharpening:upper bounds

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$\mathrm{RCA}_0 \vdash \mathrm{MDL}_{f_n}$ *for* $f_n(x) = \sqrt[A_n^{-1}(x)]{x}$.

Proof: Assume, without loss of generality, that $A_n^{-1}(i^{A_n(i)}) = i$,

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Fix $c, l > 2^c$ and k with the above property.

Sharpening:upper bounds

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$$(l + \sqrt[k]{k^{A_n(k)}})^c \le (2\sqrt[k]{k^{A_n(k)}})^c \le k^{A_n(k)}.$$

□

Thank you for listening.

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