Seetapun’s Theorem revisited

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Ramsey’s Theorem

Definition
For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all $n$-element subsets of $A$.

Theorem (Ramsey, 1930)
Suppose $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k - 1\}$. Then there is an infinite set $H \subseteq \mathbb{N}$ such that $f$ is a constant on $[H]^n$.

This version is denoted by $\text{RT}_k^n$.

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- Goal: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?

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Today we only look at

- $\text{RCA}_0$: $\text{PA}^- + \Sigma^0_1$-induction and $\Delta^0_0$-comprehension;
- $\text{WKL}_0$: $\text{RCA}_0$ and every infinite binary tree has an infinite path;
- $\text{ACA}_0$: $\text{RCA}_0$ and arithmetical comprehension: for $\varphi$ arithmetic, $\exists X \forall n (n \in X \iff \varphi(n))$.

(Subsystems in first order arithmetic has also been used, $I\Sigma_1 < B\Sigma_2 < I\Sigma_2 < \cdots$.)
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Models are often specified by their second order part:

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Since 1972, recursion theorists have been studying the “effective” content of Ramsey’s Theorem.

Basic Question: Suppose $f : \mathbb{N}^n \rightarrow \{0, \ldots, k - 1\}$ is recursive. What can we say about the complexity of infinite $f$-homogeneous sets $H$?

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Jockusch’s results (phrased in reverse math)

Theorem (1972)

1. ACA$_0$ $\vdash$ RT$_k^n$.

2. RCA$_0$ + RT$_2^3$ implies ACA$_0$.

3. WKL$_0$ does not imply RT$_2^2$. 
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Seetapun’s Theorem

Theorem (Seetapun and Slaman, 1995)
\( \text{RT}_2^2 \) is strictly weaker than \( \text{ACA}_0 \).

It revived the area after more than 20 years silence.

Basic idea: (1) avoiding the upper cone; and (2) iterate.

In this talk, I will highlight only one important ingredient, and ignore other issues like iteration and relativization issues.
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Classical Cone Avoiding: Finding Splits

Proposition
Given $C$ nonrecursive, there is a nonrecursive set $A \not\geq_T C$.

Proof (Sketch)
It is easy to make $A$ nonrecursive.

We use Cohen forcing to satisfy requirements $\Phi^A_e \neq C$ for all $e \in \omega$.

Given an initial segment $p$, ask if there are extensions $q_1$ and $q_2$ of $p$ such that for some $x$,

$$\Phi^q_{e_1}(x) \downarrow \neq \Phi^q_{e_2}(x) \downarrow.$$

(I will loosely call such $q_1$ and $q_2$ an e-split or simply a split.)

Note the question is $\Sigma^0_1$. 
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Case 1: There is a split.

Then one of the values $\Phi_{e_1}^{q_1}(x)$ and $\Phi_{e_2}^{q_2}(x)$ must disagree with $C(x)$, choose the extension which give the disagreement.

Case 2: There is no split.

Then if $\Phi_e^A$ is total and $p \subseteq A$ then $\Phi_e^A$ is recursive. $\Phi_e^A$ cannot compute the nonrecursive set $C$. 
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Then if \( \Phi^A_e \) is total and \( p \subset A \) then \( \Phi^A_e \) is recursive. \( \Phi^A_e \) cannot compute the nonrecursive set \( C \).
A Useful Decomposition

Theorem (Cholak, Jockusch and Slaman, 2001)

\[ \text{RT}_2^2 = \text{COH} + \text{SRT}_2^2. \]

- This decomposition turns out to be extremely useful.
- Thus, we can (iteratively) add a solution to COH and then a solution to SRT\(_2^2\), instead of adding a solution to RT\(_2^2\).
- (Other combinatorial principles weaker than RT\(_2^2\) can often be decomposed in a similar fashion.)
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- (Other combinatorial principles weaker than RT$_2^2$ can often be decomposed in a similar fashion.)
Let $R$ be an infinite set and $R^s = \{t|(s,t) \in R\}$. A set $G$ is said to be $R$-cohesive if for all $s$, either $G \cap R^s$ is finite or $G \cap \overline{R^s}$ is finite.

The cohesive principle COH states that for every $R$, there is an infinite $G$ that is $R$-cohesive.

We say that a coloring $f$ for pairs is stable, if for all $x$,

$$\lim_{y \to \infty} f(x, y)$$

exists.

SRT$_2$ states that every stable coloring of pairs has a solution.
**COH and SRT$_2^2$**

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An Equivalent Decomposition

- A stable coloring naturally induces a partition of $\omega$ into two $\Delta^0_2$ sets.

- Let $D^2_2$ be the statement: Every $\Delta^0_2$ set contains an infinite subsets either as a subset or as a subset of its complement.

- $RT^2_2 = D^2_2 + COH.$
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$RT^2_2 = D^2_2 + COH.$
The following is a weaker version of Jockusch and Stephan (1993)

**Theorem**

*Let $R$ be a recursive set and $C$ nonrecursive. Then there is an $R$-cohesive set $G$ with $C \not\leq_T G$.***

We use (effective) Mathias forcing.

- The conditions are pairs $(\sigma, X)$ where $\sigma$ is a finite set, $X$ is an infinite set and $\max \sigma < \min X$.
- $(\tau, Y) \leq (\sigma, X)$ if $\tau \supseteq \sigma$, $Y \subseteq X$ and $\tau \setminus \sigma \subseteq X$.

We say the forcing is recursive if the sets $X$ in the definition are recursive.
COH and cone avoiding

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Proof Sketch

Fix $R$ and $C$. The set $D_s = \{ (\sigma, X) : X \subset R^s \lor X \subset (\overline{R}^s) \}$ is dense. That settles $R$-cohesiveness.

Use the same split finding trick to do cone avoiding (just extend $\sigma$ as in the classical case).
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**Theorem**

Let $D$ be a $\Delta^0_2$ set and $C$ be a nonrecursive set. Then there is an infinite set $H$ such that either $H \subset D$ or $H \subset \overline{D}$ with $C \not\leq_T H$.

Main difficulty: When we find a split, it may involve both element in $D$ and $\overline{D}$.

Q: Is there a way to find “$D$-safe” splits?
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$D_2^2$ and cone avoiding

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Blobs and Seetapun Trees

- A sequence of *blobs* is just a recursive sequence of finite sets $\tilde{o}$ such that for each $s$ less than the length of the sequence, $\max o_s < \min o_{s+1}$.

- Let $\tilde{o}$ be a finite sequence of blobs, say of length $h$. Consider the set $T$ of all choice functions $\sigma$ with domain $h$ such that $\sigma(s) \in o_s$. $T$ can be viewed naturally as a tree, called the *Seetapun tree* associated with $\tilde{o}$.

- For a $\Sigma^0_1$-formula $\psi(G)$, we will search for blobs $o$ such that $\psi(o)$ holds.

- For example, for cone avoiding, we are looking for a finite set $o$ having a split $q_1, q_2 \subset o$. 
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Definition
Given a $\Sigma^0_1$-formula $\psi(G)$, an Seetapun disjunction for $\psi$ is a pair $(\bar{o}, S)$, where $\bar{o}$ is a sequence of blobs of length $h > 0$ and $S$ is the Seetapun tree associated with $\bar{o}$, such that:

(i) For each $s < h$, $\psi(o_s)$ holds “independently”.
(ii) For each maximal branch $\tau$ of $S$, there exists a finite subset $\iota \subseteq \tau$ such that $\psi(\iota)$ holds. We refer to the set $\iota$ as a thread (in $\tau$).

Key observation: For an Seetapun disjunction, either there is a blob $o \subseteq D$ or there is a thread $\iota \subseteq \bar{D}$. Seetapun disjunction is $D$-save!
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Proof Sketch

Fix $\Delta_2^0$ set $D$ and nonrecursive set $C$. We want to find and infinite $H \subset D$ or $\subset \overline{D}$ satisfying

$$R_{e,i} : \Phi^H_e \neq C \lor \Phi^H_i \neq C.$$  

We recursively enumerate blobs containing an $e$-split.

Case 1. This sequence of blobs is finite, i.e., after $\langle o_i : i \leq s \rangle$ there is no more $e$-splits.

Then we simply move the construction “above the last blob”. We refer this as skipping.
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Case 2. The sequence of blobs is infinite. Then we form Seetapun tree along the way and check if every branch $\tau$ contains an $i$-split.

Subcase 2.1. Every branch $\tau$ contains an $i$-split, i.e., we found an Seetapun disjunction.

Then either $D$ contains a blob $o$ or $\overline{D}$ contains a thread $\iota$. Say $D \supset o$. We choose the subset of $o$ which gives us the value $\neq C$.

Subcase 2.2. No Seetapun disjunction occurs.

Then we get an infinite subtree $T$ of the Seetapun tree. Any infinite branch will not see an $i$-split.
Proof Sketch (conti.)

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