Automatic Sequences and Transcendence of Real Numbers

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- π and e are transcendental. (Hermite 1873, Lindemann 1882)
- Is $\pi + e$ transcendental?

- e^{π} is transcendental.
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Corollary

If m and n are positive integers with $m > 1$, then $\log_m n$ is either rational or transcendental.

How about these numbers?

- π^e;
- $\pi + e$, $\pi \cdot e$ (we do know that at least one of them is transcendental);
- Euler's constant:

$$\gamma = \lim_n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right);$$

- Riemann zeta function:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots$$

  for $s > 1$.

  - when s is even, $\zeta(s)$ is a rational number times π^s, proved by Euler.
  - $\zeta(3)$ is irrational, proved by Apery, a French mathematician.
  - we do not know, when $s > 3$ is odd.

We can ask more questions on this topic.

Approximating real numbers by rationals

Definition

A real number α is approximable by rationals to order s if there exists a constant c such that the equalities

$$0 < |\alpha - \frac{p}{q}| < \frac{c}{q^s}$$

are satisfied by infinitely many pairs of integers (p, q).

We say that α is well approximable by rationals if it is approximable to a high order, and poorly approximable if it is approximable only to a low order.

Example

The number

$$\alpha = \sum_{k=0}^{\infty} 10^{-2^k} = 0.110100010000000100 0\cdots,$$

is approximable to order 2.

For any m, we let $q = 10^{2^m}$ and $p = q\sum_{k=0}^{m} 10^{-2^k}$, and then

$$0 < |\alpha - \frac{p}{q}| = \frac{1}{10^{2^{m+1}}} + \frac{1}{10^{2^{m+2}}} + \cdots < \frac{2}{10^{2^{m+1}}} = \frac{2}{q^2}.$$

Theorem

- Any real number is approximable to order 1, at least.
- Any irrational real number is approximable to order 2, at least.
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Roth's Theorem (1955)

- No algebraic real number can be approximable to any order $s > 2$.

Roth's theorem was the culmination of a series of results on approximability properties of algebraic numbers, including Liouville's, Thue's, Siegel's previous work.

Kurt Roth was awarded the Fields Medal in 1958.

Liouville numbers

Definition

A real number which is approximable to arbitrarily high order is called a Liouville number.

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000 0\cdots$$

is a Liouville number.

- $\displaystyle\sum_{k=0}^{\infty} 10^{-2^k}$ is transcendental, but not Liouville number.
- π and e are transcendental, but not Liouville number.

  Mahler proved in 1953 that for all rational numbers $\dfrac{p}{q}$ with $q \geq 2$,

$$|\pi - \frac{p}{q}| > \frac{1}{q^{42}}.$$

Thue-Morse sequence

The Thue-Morse sequence is defined inductively as follows:

$$\begin{aligned} t_0 &= 0, \\ t_{2k} &= t_k, \\ t_{2k+1} &= 1 - t_k. \end{aligned}$$

This sequence was obtained by Thue in 1906 and Morse in 1921 independently.

The Thue-Morse sequence can also be obtained in the following way:

Start from 0, and iterate the following substitution:

$$0 \longrightarrow 01, \qquad 1 \longrightarrow 10.$$

$$\begin{aligned} & 0 \\ \longrightarrow\ & 01 \\ \longrightarrow\ & 0110 \\ \longrightarrow\ & 01101001 \\ \longrightarrow\ & 0110100110010110 \\ & \cdots\cdots \end{aligned}$$

One more definition

Let $s_2(n)$ be the sum of the bits in the binary representation of n.

Then

$$t_n = s_2(n) \quad mod\ 2.$$

That is, a_n is the parity of the binary representation of n.

Taking this sequence as the digits of an infinite decimal, and we obtain the number

$$\tau = 0.110100110010110 \cdots .$$

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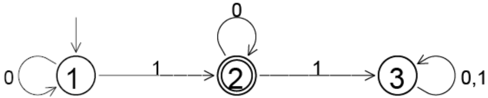
τ is irrational.

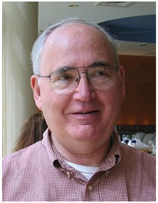
Finite automata

Here is an example of a deterministic finite automaton:



Given a DFA $\mathcal{M}$, and a integer n, we write n in binary form, start in the initial state and follow the arrows labelled with the digits of n, read from left to right.

- If after "processing" the last digit, we have arrived at an accepting state, we then say that $\mathcal{M}$ accepts n.
- Otherwise, we say that $\mathcal{M}$ rejects n.



Turing award, 1976: Citation

For their joint paper "Finite Automata and Their Decision Problem", which introduced the idea of nondeterministic machines, which has proved to be an enormously valuable concept. Their (Scott and Rabin) classic paper has been a continuous source of inspiration for subsequent work in this field.

Let

$$a_k = \begin{cases} 1 & \text{if } \mathcal{M} \text{ accepts } k; \\ 0 & \text{otherwise.} \end{cases}$$

Write down a "decimal" $a_0.a_1a_2\cdots$ in any base $b \geq 2$ and we denote it as $\alpha_{\mathcal{M}}$.

$$\begin{aligned} \alpha_{\mathcal{M}} &= a_0.a_1a_2\cdots \\ &= \sum_{k=0}^{\infty} \frac{a_k}{b^k} \\ &= \sum_{\mathcal{M} \text{ accepts } k} \frac{1}{b^k}. \end{aligned}$$

We regard $\alpha_{\mathcal{M}}$ as being the value at $\frac{1}{b}$ of the function

$$f_{\mathcal{M}}(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{\mathcal{M} \text{ accepts } k} z^k.$$

- This is the generating function of the sequence $\{a_k\}$.

Note that the numbers accepted by the DFA above are just those powers of 2. Then this automaton has the generating function

$$f(z) = \sum_{k \text{ is a power of } 2} z^k = \sum_{m=0}^{\infty} z^{2^m}.$$

Also note that

$$f(z) = f(z^2) + z.$$

When $b = 10$,

$$f(\frac{1}{10}) = \sum_{m=0}^{\infty} 10^{-2^m}$$

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In 1926, Mahler proved that for any algebraic number ζ with $0 < |\zeta| < 1$, $f(\zeta)$ is always transcendental.

How about the number τ?

Let

$$g(z) = \sum_{n=0}^{\infty} t_n z^n$$

where t_n is the n-th term in the Thue-Morse sequence.

Then

$$g(z) = (1-z)g(z^2) + \frac{z}{1-z^2}.$$

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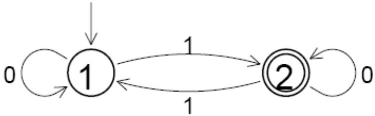
Again, for this g, when ζ is an algebraic number with $0 < |\zeta| < 1$, $g(\zeta)$ is transcendental.

This shows that τ, i.e., $g(\frac{1}{10})$, is transcendental.

This work was first proved by Kurt Mahler in 1929.

This is an automaton for the Thue-Morse sequence:



Definition

A sequence $\{a_n\}$ is k-automatic if there exists a DFA $\mathcal{M}$ accepting as input the base k expansion of n and outputs a_n.

Say that two integers $k, l \geq 2$ are multiplicatively dependent if there are two positive integers m, n such that $k^m = l^n$.

k and l are multiplicatively independent, if they are not multiplicatively dependent.

Cobham's Theorem

Proposition

- If k, l are multiplicatively dependent, then a k-automatic sequence is also l-automatic.
- If a sequence is ultimately periodic, then it is k-automatic for any $k \geq 2$.

Cobham's Theorem

Let k and l be multiplicatively independent.

If a sequence is both k-automatic and l-automatic, then this sequence is ultimately periodic.

So the Thue-Morse sequence is not 3-automatic.

Changing bases kills algebraicity

Theorem (Loxton and van der Poorten (1988), Adamczewski and Bugeaud (2007))

If the coefficients of the base-k expansion of a real number form an automatic sequence, then this number is either rational or transcendental.

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Thanks!