Probabilistic Arguments in Reverse Mathematics

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Basics of Ramsey Theory

$[X]^r$ is the set of r-element subsets of X.

A c-coloring is a function with range contained in $c = \{0, 1, \ldots, c-1\}$.

If a coloring f is constant on $[H]^r$ then H is homogeneous for f.

Theorem (Ramsey)

For every finite r and c, every $f : [\omega]^r \to c$ admits an infinite homogeneous set.

RT^r_c is the instance of Ramsey's Theorem for fixed r, c.

An Old Result

We identify a set of natural numbers with an element of 2^ω. The (Lebesgue) measure on 2^ω is induced by the following function:

$$m\{X \in 2^\omega : \sigma \prec X\} = 2^{-|\sigma|}, \sigma \in 2^{<\omega}.$$

Theorem (Sacks)

If X is not recursive then $m\{Y : X \leq_T Y\} = 0$.

Corollary (Jockusch)

There exists a recursive $f : [\omega]^3 \to 2$ s.t.

$$m\{X : (\exists Y \leq_T X)(Y \text{ is infinite and homogeneous for } f)\} = 0.$$

Proof.

There exists a recursive $f : [\omega]^3 \to 2$ s.t. every infinite f-homogeneous set computes the halting problem. □

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Homogeneous Sets for Colorings of Pairs

A coloring $f : [\omega]^2 \to c$ is stable iff $\lim_y f(x, y)$ exists for all x.

Theorem (Mileti)

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Rainbow Ramsey Theorem

A coloring $f : [\omega]^r \to \omega$ is 2-bounded iff $|f^{-1}(c)| \leq 2$ for all c.

If f is injective on $[H]^r$ then H is a rainbow for f.

RRT^r_2: every 2-bounded $f : [\omega]^r \to \omega$ admits an infinite rainbow.

Theorem (Galvin)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^r_2 \to \mathsf{RRT}^r_2$.

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Theorem (Csima and Mileti)

If $f : [\omega]^2 \to \omega$ *is recursive and* 2*-bounded then:*

$$m\{X : (\exists Y \leq_T X)(Y \text{ is an infinite rainbow for } f)\} = 1.$$

SRT^2_2: RT^2_2 for stable colorings.

Corollary (Csima and Mileti)

$\mathsf{RCA}_0 + \mathsf{RRT}^2_2 \nvdash \mathsf{SRT}^2_2$.

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RRT^2_2

A theorem of Csima and Mileti: Proof

It suffices to prove the theorem for recursive and 2-bounded f s.t.

$$(\forall (x,y),(x',y') \in [\omega]^2)(y \neq y' \to f(x,y) \neq f(x',y')).$$

Define a recursive tree $T \subset [\omega]^{<\omega}$ by induction:

1. $\emptyset \in T$;
2. Let $V(\sigma) = \{x : \sigma\langle x\rangle \text{ is a rainbow for } f\}$ for each σ. If $\sigma \in T$ and x is among the first $\min\{2^b : 2^b \geq 2^{|\sigma|+1}(|\sigma|+1)\varepsilon^{-1}\}$ many elements in $V(\sigma)$ then $\sigma\langle x\rangle \in T$.

Applying the pigeonhole principle, we can show that T is recursively isomorphic to a recursively enumerable tree $S \subseteq 2^{<\omega}$ s.t. $m[S] > 1 - \varepsilon$ and every $X \in [S]$ computes an infinite f-rainbow.

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Theorem (WW)

Every recursive 2-bounded $f : [\omega]^3 \to \omega$ admits an infinite rainbow which does not compute the halting problem.

We know that $\mathsf{RCA}_0 \vdash \mathsf{RT}^3_2 \leftrightarrow \mathsf{ACA}_0$ by Jockusch.

Corollary (WW)

$\mathsf{RCA}_0 + \mathsf{RRT}^3_2 \nvdash \mathsf{ACA}_0$.

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RRT^3_2

RRT^3_2 is weak: Proof

A set C is cohesive for $\vec{R} = (R_n : n < \omega)$ iff C is infinite and for all n either $C \cap R_n$ or $C - R_n$ is finite.

Theorem (Folklore)

Every $\vec{R}$ admits a cohesive set not computing the halting problem.

Fix a recursive 2-bounded f. We may assume that for all (x, y, z),

$$f(x, y, z) = \min\{\langle u, v, z\rangle \leq \langle x, y, z\rangle : f(u, v, z) = f(x, y, z)\}.$$

Let $\vec{R}$ be the sequence of $R_{u,v,x,y} = \{z : f(x, y, z) = \langle u, v, z\rangle\}$.

Let C be $\vec{R}$-cohesive with $K \not\leq_T C$. Define

$$\bar{f}(x, y) = \lim_{z \in C} f(x, y, z), \text{ for } (x, y) \in [C]^2.$$

Let $H \in [C]^\omega$ be an $\bar{f}$-rainbow with $K \not\leq_T C \oplus H$. Then H computes an infinite rainbow for f.

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We sketch a proof for 2-bounded f s.t.

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for all $(x, y) \in [\omega]^2$.

We need:

Theorem (Dzhafarov and Jockusch)

Every $f : \omega \to c$ for finite c admits an infinite homogeneous set which does not compute the halting problem.

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A Mathias condition is a pair $(\sigma, X) \in [\omega]^{<\omega} \times [\omega]^{\omega}$ s.t. $\max \sigma < \min X$. (σ, X) is identified with

$$\{Y \in [\omega]^{\omega} : \sigma \subset Y \subseteq \sigma \cup X\}.$$

$(\tau, Y) \leq_M (\sigma, X)$ iff $(\tau, Y) \subseteq (\sigma, X)$.

(σ, X) is admissible iff

1. X does not compute the halting problem;
2. $\sigma\langle x\rangle$ is an f-rainbow for all $x \in X$.

We shall build a descending (w.r.t. $\leq_M$) sequence $((\sigma_n, X_n) : n < \omega)$ of admissible Mathias conditions and get the desired f-rainbow $G = \bigcup_n \sigma_n$.

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Let $\mathcal{F}$ be the set of 2-bounded colorings g s.t.

$$g(x,y) = \min\{\langle u,y\rangle \leq \langle x,y\rangle : g(u,y) = g(x,y)\}$$

for all $(x,y) \in [\omega]^2$. Then $\mathcal{F}$ is Π^0_1 and compact.

For each (σ, X), let $\mathcal{F}_{\sigma,X}$ be the set of $g \in \mathcal{F}$ s.t. $\sigma\langle x\rangle$ is a g-rainbow for each $x \in X$. Then $\mathcal{F}_{\sigma,X}$ is Π^X_1.

If (σ, X) is admissible then $f \in \mathcal{F}_{\sigma,X}$.

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Rainbows for Arbitrary Colorings of Pairs

A probability measure

For each finite tree $T \subset [\omega]^{<\omega}$, let

$$[T] = \{\sigma \in T : (\forall x)(\sigma\langle x\rangle \notin T)\} = \text{ the set of leaves of } T.$$

We define

$$m_T(\emptyset) = 1,$$

$$m_T(\sigma\langle x\rangle) = \frac{m_T(\sigma)}{|\{y : \sigma\langle y\rangle \in T\}|}, \text{ if } \sigma\langle x\rangle \in T.$$

If $S \subseteq [T]$ then

$$m_T S = \sum_{\sigma \in S} m_T(\sigma).$$

We write $(P_T\sigma > \varepsilon)\varphi$ for $m_T\{\sigma \in [T] : \varphi(\tau)\} > \varepsilon$ etc.

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Trees of rainbows

For $g \in \mathcal{F}_{\sigma,X}$, let $\mathcal{T}(\sigma, X, g)$ be the set of finite trees $T \subset [\omega]^{<\omega}$ s.t.

1. $\sigma\tau$ is a g-rainbow for each $\tau \in T$;
2. If $\tau \in T$ is not a leaf then

$$|\{x : \tau\langle x\rangle \in T\}| \geq \min\{2^b : 2^b \geq 2^{|\tau|+3}|\sigma\tau|\}.$$

Lemma

If $T \in \mathcal{T}(\sigma, X, g)$ then

$$(\forall^\infty x \in X)(P_T\tau \geq \frac{3}{4})(\sigma\tau\langle x\rangle \text{ is a rainbow for } g)$$

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Extending an admissible condition

Lemma

Every admissible (σ, X) *can be extended to an admissible* (τ, Y) *with* $|\tau| > |\sigma|$.

Let $T \subseteq [X]^{\leq 1}$ be a tree in $\mathcal{T}(\sigma, X, f)$. For sufficiently large $x \in X$, let

$$h(x) = \{\xi \in [T] : \sigma\xi\langle x\rangle \text{ is not a rainbow for } f\}.$$

Then $m_T h(x) \leq \frac{1}{4}$ for sufficiently large $x \in X$.

By the theorem of Dzhafarov and Jockusch, pick $Y \in [X]^\omega$ and $\xi \in [T]$ s.t. Y is h-homogeneous, $K \not\leq_T Y$ and $\xi \notin h(Y)$.

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Lemma

For each admissible (σ, X) and e there exists an admissible $(\tau, Y) \leq_M (\sigma, X)$ s.t. $K \neq \Phi_e(Z)$ for all $Z \in (\tau, Y)$.

Our plan is to follow Seetapun's trick:

1. As f is of arbitrary complexity, we cannot directly consult f when finding (τ, Y).
2. Instead of asking questions about f, we ask questions like: whether $\mathcal{F}_{\sigma,X}$ contains some element satisfying certain Π^0_1 property (say φ).
3. If $\mathcal{F}_{\sigma,X}$ does contain such an element, then we can pick $g \in \mathcal{F}_{\sigma,X}$ not computing K. With g, we can obtain an admissible extension which forces some Π^0_1 statement $(\Phi_e(Z; x)\uparrow)$.
4. Otherwise, f is a particular element of $\mathcal{F}_{\sigma,X}$ which satisfies a Σ^0_1 property $(\neg\varphi)$. From this fact, we can extend (σ, X) in a finitary way to force a Σ^0_1 statement.

Rainbows for Arbitrary Colorings of Pairs

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A pair (η_0, η_1) *e-splits over σ* iff

$$(\exists x)(\Phi_e(\sigma\eta_0; x)\downarrow \neq \Phi_e(\sigma\eta_1; x)\downarrow).$$

Let $\mathcal{U}$ be the set of $g \in \mathcal{F}_{\sigma,X}$ s.t.

$$(\forall T \in \mathcal{T}(\sigma, X, g))(P_T\tau < \frac{1}{2})(\tau \text{ contains a pair } e\text{-splitting over } \sigma)$$

Then $\mathcal{U} \in \Pi^X_1$.

Rainbows for Arbitrary Colorings of Pairs

The key lemma: splittings

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Rainbows for Arbitrary Colorings of Pairs

The key lemma: Case 1

Case 1: $\mathcal{U} \neq \emptyset$.

Let $g \in \mathcal{U}$ be s.t. $K \not\leq_T X \oplus g$. By relativizing the theorem of Csima and Mileti, let $R \in [X]^\omega$ be s.t. R is a g-rainbow and $K \not\leq_T X \oplus R$.

We can build a $g \oplus R$-recursive tree T s.t. $T_l \in \mathcal{T}(\sigma, R, g)$ and every leaf of T_l is of length l where $T_l = T \cap [R]^{\leq l}$. Let

$$S = \{\tau \in T : \tau \text{ contains no pair } e\text{-splitting over } \sigma\}.$$

Then $(P_{T_l}\tau > \frac{1}{2})(\exists Y \in [S])(\tau \prec Y)$ for all l.

So we can pick $Y \in [S]$ s.t. $K \not\leq_T Y$.

As (σ, X) is admissible and $Y \subseteq X$, (σ, Y) is admissible.

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Rainbows for Arbitrary Colorings of Pairs

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The Hierarchy of Rainbow Ramsey Theorem

Theorem (Csima and Mileti)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{RRT}^3_2$. *Consequently,* $\mathsf{RCA}_0 + \mathsf{RRT}^2_2 \nvdash \mathsf{RRT}^3_2$.

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$\mathsf{RCA}_0 + \mathsf{RRT}^3_2 \nvdash \mathsf{RRT}^4_2$.

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Triples Jumps of Rainbows

A set X is low$_n$ iff $X^{(n)} \equiv_T \emptyset^{(n)}$.

Theorem (WW, Triple Jump Theorem)

Every $\emptyset'$-recursive 2-bounded coloring of pairs admits a low$_3$ *infinite rainbow G.*

With the above theorem, we can separate RRT^3_2 and RRT^4_2.

Proof of $\mathrm{RCA}_0 + \mathrm{RRT}^3_2 \nvdash \mathrm{RRT}^4_2$.

By Cholak, Jockusch and Slaman, every recursive $\vec{R}$ admits a low$_2$ cohesive set. Combine this and the triple jump theorem, we get an ω-model $\mathcal{M} \models \mathrm{RCA}_0 + \mathrm{RRT}^3_2$ which is bounded by $\emptyset^{(3)}$.

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Theorem (WW)

Every $\emptyset'$-recursive $\vec{R}$ admits a low_3 *cohesive set.*

Then we can reduce the triple jump theorem to:

Theorem (WW, Double Jump Theorem)

If f is a $\emptyset'$-recursive 2-bounded coloring of pairs s.t.

$$(\forall(x,y))(f(x,y) = \min\{\langle u,y\rangle : f(u,y) = f(x,y)\})$$

and $\lim_y f(x,y)$ exists for all x, then there exists a low_2 *infinite f-rainbow.*

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Theorem (WW, Double Jump Theorem)

If f is a $\emptyset'$-recursive 2-bounded coloring of pairs s.t.

$$(\forall(x,y))(f(x,y) = \min\{\langle u,y\rangle : f(u,y) = f(x,y)\})$$

and $\lim_y f(x,y)$ exists for all x, then there exists a low_2 infinite f-rainbow.

Double Jump Theorem

The forcing notion

We fix f as in the Double Jump Theorem.

Recall that $\mathcal{F}_{\sigma,X}$ is the set of 2-bounded g of pairs s.t.

$$(\forall(x,y))(g(x,y) = \min\{\langle u,y\rangle : g(u,y) = g(x,y)\})$$

and $\sigma\langle x\rangle$ is a g-rainbow for all $x \in X$.

We build a desired rainbow G by forcing with conditions $(\sigma, X, \vec{g})$ s.t. (σ, X) is a Mathias condition with $f \in \mathcal{F}_{\sigma,X}$, $\vec{g}$ is a finite sequence of colorings in $\mathcal{F}_{\sigma,X}$ and $X \oplus \vec{g}$ is low.

$(\tau, Y, \vec{h}) \leq (\sigma, X, \vec{g})$ iff $(\tau, Y) \leq_M (\sigma, X)$ and $\vec{g} \subseteq \vec{h}$.

A $\vec{g}$-rainbow is a set which is a rainbow for every $g \in \vec{g}$. A condition $(\sigma, X, \vec{g})$ is meant to represent the following set:

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Double Jump Theorem

Fast growing trees

Let $\mathcal{T}(n, a, b)$ be the set of all finite trees $T \subset [\omega]^{<\omega}$ s.t.

$$\sigma \in T - [T] \rightarrow |\{x : \sigma\langle x\rangle \in T\}| \geq 2^{(n+1)(n+|\sigma|+1)+b+1}(a + |\sigma| + 1)^b.$$

Lemma (Fast Growing Lemma)

Let $T \in \mathcal{T}(n + m, a, b)$ and $P \subseteq [T]$ be s.t. $m_T P \geq 2^{-m}$. Then there exists $S \in \mathcal{T}(n, a, b)$ s.t. $[S] \subseteq P$ and $m_T(P - [S]) < 2^{-m}$.

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Fast growing trees of rainbows

Let $\mathcal{T}_R^X(n,\sigma,\vec{g})$ be the set of $T \in \mathcal{T}(n,|\sigma|,|\vec{g}|)$ s.t.

$$T \subset [X]^{<\omega} \wedge (\forall \tau \in T)(\sigma\tau \text{ is a rainbow for } \vec{g}).$$

Lemma

Suppose that (σ, X) is a Mathias condition, $\vec{g}$ is a finite sequence from $\mathcal{F}_{\sigma,X}$ and $T \in \mathcal{T}_R^X(n,\sigma,\vec{g})$. Then

$$(\forall^\infty x \in X)(P_T\tau > 1-2^{-n})(\sigma\tau\langle x\rangle \text{ is a rainbow for } \vec{g}).$$

Hence

$$(P_T\tau > 1-2^{-n})(\exists^\infty x \in X)(\sigma\tau\langle x\rangle \text{ is a rainbow for } \vec{g}).$$

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Double Jump Theorem

Deciding one instance of Double jump: motivation

Deciding G'' is equivalent to deciding whether $\Phi_e(G)$ is total for each e. Given a condition $p = (\sigma, X, \vec{g})$, we are to extend it to a condition that decides the totality of $\Phi_e(G)$ for some e.

To decide the totality of $\Phi_e(G)$, we need an answer for: whether there are $q = (\sigma\tau, Y, \vec{g}\vec{h}) \le p$ and x s.t. q forces $\Phi_e(G; x)\uparrow$. To get such a q, we need τ which can be extended infinite often (i.e., we can get Y infinite). But the existence of such τ is too complicated.

To walk around this complicated question, instead we ask: whether for some x there is a large probability to find a suitable q.

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To walk around this complicated question, instead we ask: whether for some x there is a large probability to find a suitable q.

Double Jump Theorem

Deciding one instance of Double jump: a test

p $= (\sigma, X, \vec{g})$ passes the e-test at y, if there exist $S \in \mathcal{T}_R^X(n+6, \sigma, f\vec{g})$, m, large c and $\vec{h} \subset \mathcal{F}_{\sigma, X\cap(c,\infty)}$ such that $m \geq n+6$,

$$\forall z \in X \cap (c,\infty)(P_S\tau > 1 - 2^{-d-4})(\sigma\tau\langle z\rangle \text{ is a } \vec{g}\vec{h}\text{-rainbow})$$

(It is very likely that a leaf of S gives us an extension of p.)

and

$$(P_S\tau \geq 2^{-1})\forall T \in \mathcal{T}_R^{X\cap(c,\infty)}(m+|\tau|, \sigma\tau, \vec{g}\vec{h})(P_T\rho > 2^{-1})\Phi_e(\sigma\tau\rho; x)\uparrow.$$

(Very likely that extending σ by a leaf of S will likely force $\Phi_e(G;x)\uparrow$.)

It is uniformly $(\vec{g} \oplus X)''$-decidable whether $(\sigma, X, \vec{g})$ passes the e-test at y, if $(\sigma, X, \vec{g})$ has some good property.

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Double Jump Theorem

Large conditions

A condition $(\sigma, X, \vec{g})$ is $(\vec{e}_0, \vec{e}_1)$-large at $\vec{x}$ (where $|\vec{e}_0| = |\vec{x}|$) iff there exists a largeness witness (n, d) s.t.

(L1) for all $S \in \mathcal{T}_R^X(n, \sigma, \vec{g})$, $(P_S\tau > 2^{-1})\Phi_{\vec{e}_0}(\sigma\tau; \vec{x})\uparrow$;

(L2) for all $S \in \mathcal{T}_R^X(n, \sigma, f\vec{g})$, $m \geq n$, x, large c and $\vec{h} \subset \mathcal{F}_{\sigma,X}$, if

$$(\forall y \in Y)(P_S\tau > 1 - 2^{-d})(\sigma\tau\langle y\rangle \text{ is a rainbow for } \vec{g}\vec{h})$$

where $Y = X \cap (c, \infty)$, then

$$(P_S\tau > 2^{-1})(\exists T \in \mathcal{T}_R^Y(m + |\tau|, \sigma\tau, \vec{g}\vec{h})) \\ (P_T\rho > 2^{-1})(\forall e \in \vec{e}_1)\Phi_e(\sigma\tau\rho; x)\downarrow.$$

If (n, d) is known then being large is a Π^0_2 property.

Double Jump Theorem

Large conditions: Intuition for (L1)

(L1) For all $S \in \mathcal{T}_R^X(n, \sigma, \vec{g})$, $(P_S \tau > 2^{-1})\Phi_{\vec{e}_0}(\sigma\tau; \vec{x})\uparrow$;

$\Phi_{\vec{e}_0}(\sigma\tau; \vec{x})\uparrow$: if e is the i-th number of $\vec{e}_0$ and x is the i-th number of $\vec{x}$ for $i < |\vec{e}_0| = |\vec{x}|$ then $\Phi_e(\sigma\tau; x)\uparrow$.

By the Fast Growing Lemma, this means that for G almost everywhere in (σ, X) if G is a rainbow for $\vec{g}$ then $\Phi_e(G)$ is partial for every $e \in \vec{e}_0$.

So if G is built by a descending sequence of large conditions then some Σ^0_2 properties of G are forced by conditions in the sequence.

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Double Jump Theorem

Large conditions: Intuition for (L2)

(L2) for all $S \in \mathcal{T}_R^X(n, \sigma, f\vec{g})$, $m \geq n$, x, large c and $\vec{h} \subset \mathcal{F}_{\sigma,X}$, if

$$(\forall y \in Y)(P_S\tau > 1 - 2^{-d})(\sigma\tau\langle y\rangle \text{ is a rainbow for } \vec{g}\vec{h})$$

where $Y = X \cap (c, \infty)$, then

$$(P_S\tau > 2^{-1})(\exists T \in \mathcal{T}_R^Y(m + |\tau|, \sigma\tau, \vec{g}\vec{h}))(P_T\rho > 2^{-1})(\forall e \in \vec{e}_1)\Phi_e(\sigma\tau\rho; x)\downarrow.$$

The first part means that: for a large probability, $\tau \in [S]$ and $\vec{h}$ give us something like an extension $(\sigma\tau, Y, \vec{g}\vec{h})$ of $(\sigma, X, \vec{g})$.

The second part means that: for a large probability, $\sigma\tau$ admits many extensions $\sigma\tau\rho$ s.t. $\sigma\tau\rho$ is a $\vec{g}$-rainbow and $\Phi_e(\sigma\tau\rho; x)\downarrow$.

Together: if we can find sufficiently many extensions of $(\sigma, X, \vec{g})$ then we have a good chance to extend some to approximate the Π_2^0 properties of G indexed by $\vec{e}_1$.

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Together: if we can find sufficiently many extensions of $(\sigma, X, \vec{g})$ then we have a good chance to extend some to approximate the Π_2^0 properties of G indexed by $\vec{e}_1$.

Double Jump Theorem

Approximating Π^0_2 properties

Lemma

If a condition $p = (\sigma, X, \vec{g})$ is $(\vec{e}_0, \vec{e}_1)$-large at $\vec{x}$, then for every x there exists $q = (\tau, Y, \vec{g}) \leq p$ s.t. q is $(\vec{e}_0, \vec{e}_1)$-large at $\vec{x}$ and $\Phi_e(\tau; x)\downarrow$ for all $e \in \vec{e}_1$.

Moreover, τ and a lowness index of $\vec{g} \oplus Y$ can be obtained from $(\sigma, x, \vec{e}_0, \vec{e}_1, \vec{x})$ and a lowness index of $\vec{g} \oplus X$, in a uniformly $\emptyset''$-recursive way.

Double Jump Theorem

Deciding new Σ^0_2/Π^0_2 properties

Lemma

Let $(\sigma, X, \vec{g})$ be $(\vec{e}_0, \vec{e}_1)$-large at $\vec{x}$. For each e, one of the followings holds:

(1) *there exist y and $q = (\tau, Y, \vec{h}) \le (\sigma, X, \vec{g})$ s.t. q is $(\vec{e}_0\langle e\rangle, \vec{e}_1)$-large at $\vec{x}\langle y\rangle$; (so a new Σ^0_2 property is forced)*

(2) *$(\sigma, X, \vec{g})$ is $(\vec{e}_0, \vec{e}_1\langle e\rangle)$-large at $\vec{x}$. (so a new Π^0_2 property is forced)*

Moreover, it is $\emptyset''$-decidable whether (1) or (2) holds; and in (1), τ and a lowness index of $\vec{h} \oplus Y$ can be obtained from $(\sigma, \vec{e}_0, \vec{e}_1, \vec{x}, e, y)$ and a lowness index of $\vec{g} \oplus X$, in a uniformly $\emptyset''$-recursive manner.

The effectiveness here is by the effectiveness of the test that we introduced before.

Questions

1. It is shown that RRT^r_2 is always strictly weaker than ACA_0. Do RRT^r_2's for positive r's give us a proper hierarchy of combinatorial principles below ACA_0?
2. Recent development shows that there are many other consequences $(\mathsf{FS}^r, \mathsf{TS}^r)$ of RT^r_2 strictly weaker than ACA_0. Do they give rise to some proper hierarchies below ACA_0?
3. If f is a recursive 2-bounded coloring of $[\omega]^r$, does f admit a low_r infinite rainbow? (True for $r = 2, 3$)
4. Does every $\emptyset^{(n)}$-recursive $\vec{R}$ admit a low_{n+2} cohesive set? (True for $n = 0, 1$)
5. Does every $\emptyset^{(n)}$-recursive finite partition of ω admit a low_{n+1} infinite homogeneous set? (True for $n = 0, 1$)

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4. Does every $\emptyset^{(n)}$-recursive $\vec{R}$ admit a low_{n+2} cohesive set? (True for $n = 0, 1$)
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Mathias forcing and randomness

Theorem (WW)

If $\vec{R}$ is a recursive sequence admitting no recursive cohesive set, then $\vec{R}$ admits no cohesive set recursive in sufficiently random (Cohen generic) oracles.

Roughly, Mathias forcing and random (Cohen) forcing are incompatible.

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To develop an effective measure theory (of Baire space) compatible with Mathias forcing.

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