A survey of recent work on arithmetical equivalence relations

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Motivating questions

- Study the complexity of equivalence relations (on natural numbers) and how they interact with Turing degrees.
- As in the study of algebraic structures, investigate how to code information into structures.
- How do we compare the complexity of two ERs?
- How else can we compare? Isomorphisms and categoricity.

Precursor

- ERs are well studied in Borel theory.
- (Friedman-Stanley) Introduced the notion of Borel reducibility to compare arbitrary ERs on Borel spaces (classification problems in math, finding invariants).
- To study this in classical recursion theory, we consider ERs on ω. (Can code many things).
- Define the complexity of an equivalence relation R to be the complexity of R as a set of pairs.

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Other related work

- Fokina, Friedman study this for Σ^1_1 ERs, and hyperarithmetical reductions.
- Various authors (Fokina, Friedman, Harizanov, Knight, McCoy, Montalbán) used similar ideas to study computable structures.
- We'll look at low level (arithmetical) ERs and restrict ourselves to computable reducibilities.
- Motivation drawn from Borel theory (while not directly related). In the low level setting, things turn out to be very different.

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Arithmetical ERs and computable reducibilities

- (Bernadi, Sorbi) positive ERs
- (Fokina, Friedman) computable reducibilities for Σ^1_1 ERs
- (Gao, Gerdes) systematic study of c.e. ERs
- (Coskey, Hamkins, Miller) comparing standard ERs
- (Andrews, Lempp, Miller, N, San Mauro, Sorbi) more on c.e. ERs
- (Ianovski, Miller, Nies, N, Stephan) completeness for ERs
- (Miller, N) finitary reducibilities
- (Calvert, Cenzer, Harizanov, Morozov; Cenzer, Harizanov, Remmel) categoricity of c.e. and Π^0_1 ERs
- (Downey, Melnikov, N) $0'$-categorical ERs and Turing degrees. Also studied the applications of equivalence relations on a certain class of abelian p-groups.

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Brief history

- The study of positive (or c.e.) ERs traces back to the theory of positive numberings.
- Recall that a numbering is a pair (ν, S) where $\nu : \omega \mapsto S$ is onto.
- Numberings are ERs in disguise:
  - Given a numbering (ν, S), we can get xRy iff $\nu(x) = \nu(y)$.
  - Conversely we can get a numbering by letting all elements of each equiv class $[x]$ number the same object.
- A positive numbering is simply a numbering where the induced ER is c.e.

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Brief history

- Malcev first and later, Ershov studied systematically *positive* ERs (c.e. ERs).

Definition (Malcev)

A c.e. ER R is *precomplete* if for every partial recursive φ there is a total computable function f such that for every n,

$$\varphi(n)\downarrow \ \Rightarrow \varphi(n) \ R \ f(n)$$

f is called a totalizer.

Brief history

- The most common (natural?) way of comparing ERs is to say that $R \leq S$ iff there is a computable function f such that

$$x \; R \; y \Leftrightarrow f(x) \; R \; f(y)$$

- Ershov introduced this when considering monomorphisms in the category of all numberings.
- Analogue to the study of Borel equivalence classes, where f is a Borel function.
- Many authors study this reducibility, all under different names!
  - Bernardi, Sorbi; Gao, Gerdes: m-reducibility,
  - Fokina, Friedman: FF-reducibility,
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C.e. ERs

Definition (Bernadi, Sorbi)

A c.e. ER U is *universal* if for every c.e. ER S, we have $S \leq U$.

- Clearly, there are universal c.e. ERs.
- (Bernadi, Sorbi) Every precomplete c.e. ER is universal (but not conversely).

C.e. ERs

Some easy facts about the poset of c.e. ERs:

1. There is a greatest element (any universal c.e. ER) and a least element ($\equiv_1$).
2. There is an initial segment of type $\omega + 1$:

$$\equiv_1 \ < \ \equiv_2 \ < \ \equiv_3 \ < \ \cdots \ < \ Id$$

3. This completely describes the degrees of computable ERs. The non-computable c.e. ERs are not below this chain.

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C.e. ERs

4. We can embed the c.e. 1-degrees into the poset of c.e. ERs, by taking
$$A \mapsto R_A$$
where $x \; R_A \; y$ iff $x, y \in A$.
For instance, if A is simple then $Id \not\leq R_A$.
5. The c.e. ER is neither an upper- nor a lower-semilattice.
6. The Π^0_3 theory is undecidable.
7. The greatest element is join irreducible. (You get a problem if you consider the "natural" join operation).
8. The c.e. ER degrees is upwards dense.

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C.e. ERs

- To study the structure of c.e. ERs, Gao and Gerdes introduced a jump operator

Definition (Gao, Gerdes)

Let E be a c.e. ER. The jump of E, written as E' is defined

$$x\ E'\ y \Leftrightarrow\ \varphi_x(x)\downarrow \text{and } \varphi_y(y)\downarrow \text{and } \varphi_x(x)\ E\ \varphi_y(y).$$

- For example, the jump of the smallest element, $(=_1)' = R_K$.
- $(Id)'$ is the c.e. ER yielding the partition $\{K_i : i \in \omega\} \cup \{\{x\} : x \notin K\}$, where $K_i = \{e : \varphi_e(e)\downarrow = i\}$.

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C.e. ERs

Theorem (Gao, Gerdes)

1. $R \leq R'$
2. $S \leq R$ iff $S' \leq R'$
3. *If R is not universal then R' is not universal.*

- Clearly if R is universal then $R' \equiv R$. Is there a non-universal ER with this property?

C.e. ERs

Theorem (Andrews, Lempp, Miller, N, Sorbi)

Let E be a c.e. ER. If $E' \leq E$ then E is universal.

Corollary

The c.e. ERs is upwards dense.

C.e. ERs

- The universal c.e. ERs are exactly the ones closed under the jump. Look at notable subclasses.
- Recall each precomplete c.e. ER is universal.
- Effectively inseparable sets play a crucial role in the study of c.e. sets. Visser, Bernadi study this for ERs.
- A c.e. ER is effectively inseparable if it yields a partition into effectively inseparable sets.
- A c.e. ER is uniformly effectively inseparable if one can uniformly get a production function.

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C.e. ERs

Theorem (Andrews, Lempp, Miller, N, San Mauro, Sorbi)

1. *Each precomplete ER is uniformly effectively inseparable.*
2. *Each uniformly effectively inseparable ER is universal (and of course, effectively inseparable).*
3. *Universality and effective inseparability do not imply each other.*

- It was also shown that u.e.i. coincides with a number of previously studied notions in Bernadi, Sorbi.

Natural arithmetical ERs

- Arithmetical ERs.
- Coskey, Hamkins and Miller studied ERs based on c.e. analogues of the standard Borel relations.
- The well-studied ERs in Borel study are:
  - $E_1 = \{(A, B) : \forall^\infty n\, (A_n = B_n)\}$
  - $E_3 = \{(A, B) : \forall n\, (A_n =^* B_n)\}$
  - $E_{set} = \{(A, B) : \{A_n\} = \{B_n\}\}$
  - $Z_0 = \{(A, B) \mid \lim_n \frac{|(A \triangle B) \upharpoonright n|}{n} = 0\}$

Natural arithmetical ERs

- They considered the c.e. analogues of these relations, and showed that the situation there is different.

Theorem (Coskey, Hamkins, Miller)

$E^{ce}_{=^*} \equiv E^{ce}_1$, *where* $E^{ce}_1 = \{(A, B) : \forall^\infty n\, (A_n = B_n)\}$.

Theorem (Miller, N)

- $E^{ce}_3 \equiv Z^{ce}_0$.
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Natural arithmetical ERs

- To study naturally arising (low-level) arithmetical ERs, Coskey, Hamkins and Miller considered:

$$E^{ce}_{\min} = \{(W, V) : \min W = \min V\}$$

$$E^{ce}_{\max} = \{(W, V) : \max W = \max V\}$$

- These are Π^0_2 relations, and in fact:

Theorem (Coskey, Hamkins, Miller)

$E^{ce}_{\max}$ and $E^{ce}_{\min}$ are incomparable and below $E^{ce}_{=}$.

Proof.

If $E^{ce}_{\max} \leq E^{ce}_{\min}$ via f, we build (by the Recursion Theorem) W_i and W_j and watch $W_{f(i)}$ and $W_{f(j)}$. □

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Universal arithmetical ERs

- We've seen several examples of naturally occurring arithmetical ERs and tried to classify them.
- One can also look at algebraic structures known to have simple isomorphism problems.
- Let's instead look at the general theory – universality.
- For c.e. ERs, we've seen that this yields a rich theory (jump operator, u.e.i.).
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Universal arithmetical ERs

- By putting together all c.e. ERs, we can obtain a universal c.e. ER. Relativize this to get a universal Σ^0_n ER for each n.
- Doing this does not work to produce a universal Π^0_1 ER.
- The transitive closure of a c.e. set of pairs is c.e., but not for Π^0_1 sets of pairs. Nevertheless,

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Surprisingly, we found that:

Theorem (Ianovski, Miller, Nies, N)

For any $n \geq 2$ there is no universal Π^0_n ER.

Theorem (Fokina, Friedman and Nies)

$\{(W, V) : W \equiv_1 V\}$ and $\{(W, V) : W \equiv_m V\}$ are universal at the Σ^0_3 level.

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Another reducibility

- The usual reducibility for comparing ERs,

$$R \leq S \Leftrightarrow \exists f \forall x, y (x \; R \; y \Leftrightarrow f(x) \; S \; f(y))$$

  is sometimes too uniform.
- For instance, lack of universal ERs at Π_{n+2} levels.
- Often, when one wants to show $R \leq S$, one often first tries a "non-uniform" map.

Definition (Miller, N)

We say that R is n-arily reducible to S, and write $R \leq^n S$, if there are total computable functions $f_1, \cdots, f_n : \omega^n \mapsto \omega$, such that for all $j, k \leq n$ and all n-tuple of numbers $i_1, \cdots, i_n$, we have

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Finitary reducibility

- For example, $R \leq^2 S$ iff there are computable functions f, g such that for all pairs x, y,

$$x \; R \; y \Leftrightarrow f(x,y) \; S \; g(x,y)$$

- This seems a good alternative way to measure reducibility for ERs:

Theorem (Miller,N)

- *Equality of c.e. sets is universal at the Π^0_2 level for $\leq^n$ for all $n \geq 2$.*
- *Relativizing, we get universal ERs at the Π^0_k for every k, with respect to finitary reducibilities.*
- *E^{ce}_{max} is universal at the Π^0_2 level for $\leq^3$ (but not universal for $\leq^4$).*

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Questions

- Are there natural examples of ERs separating $\leq^{n}$ from $\leq^{n+1}$?
- Understand the structure of the partial order for Σ^0_k ERs under both reducibilities.
- Find ERs arising in algebra and fit it in the general theory.

- Thank you.