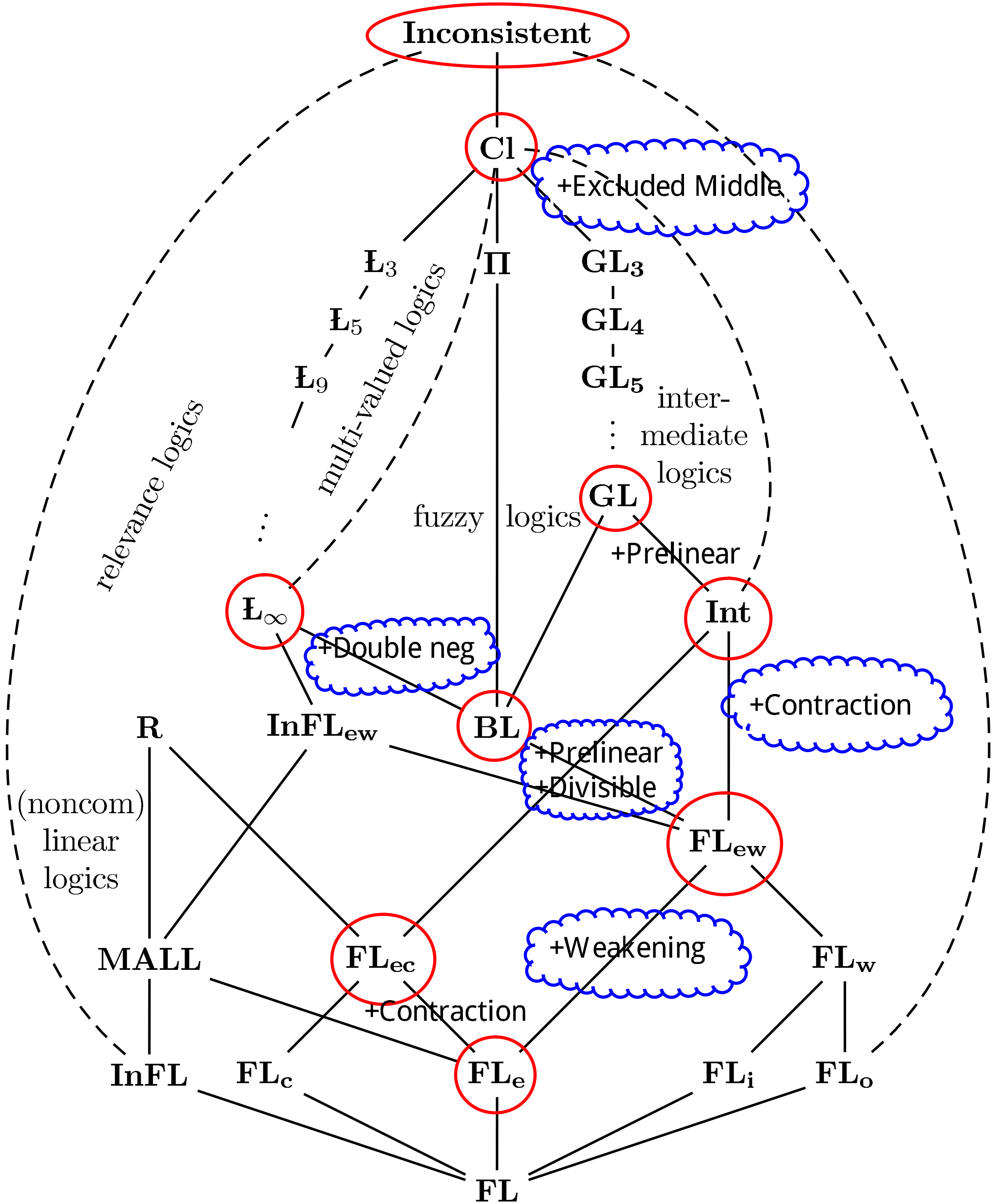
Herbrand's Theorem for Substructural Logics: from an Algebraic Perspective

Kazushige Terui

RIMS, Kyoto University

23/02/13, Sendai Logic Seminar



From
N. Galatos, P. Jipsen,
T. Kowalski and H. Ono,

Residuated Lattices:
An Algebraic Glimpse
at Substructural Logics, 2007.

Outline

A guided tour in substructural logics

Lattice: $\mathbf{L} = \langle L, \wedge, \vee \rangle$ such that
$a \leq b \Leftrightarrow a \vee b = b$ defines a partial order and

$$\frac{a \leq c \text{ and } b \leq c}{a \vee b \leq c} \qquad \frac{c \leq a \text{ and } c \leq b}{c \leq a \wedge b}$$

Residuals: given $\mathbf{L} = \langle L, \wedge, \vee \rangle$, $\mathbf{M} = \langle M, \wedge, \vee \rangle$,
$g : \mathbf{M} \longrightarrow \mathbf{L}$ is a residual of $f : \mathbf{L} \longrightarrow \mathbf{M}$ if

$$\frac{f(a) \leq x}{a \leq g(x).}$$

Fact

f is join-preserving and g is meet-preserving.

We want to define $\to$ as a residual of some operation $*$:

$$\frac{a * x \leq y}{x \leq a \to y}$$

If we take $* = \wedge$ in (bounded) $\mathbf{L} = \langle L, \wedge, \vee \rangle$, we obtain a Heyting algebra.

Proposition

A lattice $\mathbf{L}$ embeds into a Heyting algebra iff $\mathbf{L}$ is distributive:

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$$

Hence does not work for nondistributive $\mathbf{L}$.

Monoid: Let $\cdot$ be an associative operation on L:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

and $\backslash, /$ be its residuals (on the respective arguments):

$$\frac{a \cdot x \leq y}{x \leq a \backslash y} \qquad \frac{x \cdot a \leq y}{x \leq y / a}$$

Assuming $\cdot$ has the unit 1, $\leq$ can be internalized:

$$\begin{array}{c} a \leq b \\ \hline\hline a \cdot 1 \leq b \\ \hline\hline 1 \leq a \backslash b \end{array} \qquad \begin{array}{c} a \leq b \\ \hline\hline 1 \cdot a \leq b \\ \hline\hline 1 \leq b / a \end{array}$$

Residuated lattice: $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ such that

- □ $\langle A, \wedge, \vee \rangle$ is a lattice;
- □ $\langle A, \cdot, 1 \rangle$ is a monoid;
- □ $a \cdot b \leq c \;\Leftrightarrow\; b \leq a \backslash c \;\Leftrightarrow a \leq c/b$.

$\mathbf{A}$ is bounded if $\top, \bot \in A$.

An FL-algebra is a RL with constant 0: $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0 \rangle$

0 is used to define negations:

$$-a = a \backslash 0, \qquad \sim a = 0/a.$$

Some Identities

We have:

- $a \cdot (a \backslash b) \leq b$
- $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$
- $a \backslash (b \wedge c) = (a \backslash b) \wedge (a \backslash c)$
- $(a \vee b) \backslash c = (a \backslash c) \wedge (b \backslash c)$

In addition:

- $a \backslash b = b / a = a \to b$ (if $\cdot$ is commutative)
- $a \cdot b \leq a \wedge b$ (if $x \leq 1$)
- $a \cdot b \geq a \wedge b$ (if $x \leq x \cdot x$)

Proposition

Every lattice embeds into a RL. So does every monoid.

Examples:

1. *ℓ-groups:* Given a lattice-ordered group $\mathbf{G} = \langle G, \wedge, \vee, \cdot, (\ \ )^{-1}, 1 \rangle$,

$$a \backslash b = a^{-1} b, \qquad b/a = ba^{-1}$$

defines a RL.

Fact

ℓ-group is trivial iff bounded.

2. Relation algebras: Given a set X,

$$\mathbf{Rel}(X) = \langle \mathcal{P}(X^2), \cap, \cup, \circ, \backslash, /, \Delta \rangle,$$

where

$$\begin{aligned} x(R\backslash S)y &\iff \forall z(zRx \Rightarrow zSy) \\ x(S/R)y &\iff \forall z(yRz \Rightarrow xSz) \end{aligned}$$

defines a RL.

3. Ideal lattices of rings: Given a ring R,

$$\mathbf{Idl}(R) = \langle 2Idl(R), \cap, +, \cdot, \backslash, /, R \rangle$$

is an (integral) RL.

4. Powersets of monoids: Given a monoid $\mathbf{M}$,

$$\mathcal{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle, \quad \text{where}$$

$$\begin{array}{rcl} \alpha\backslash\beta & = & \{b : \forall a \in \alpha \quad a \cdot b \in \beta\} \\ \beta/\alpha & = & \{b : \forall a \in \alpha \quad b \cdot a \in \beta\} \end{array} \quad \text{is a RL.}$$

- $T(X)$:= the set of terms (formulas) over $\{\wedge, \vee, \cdot, \backslash, /, 1, 0\}$ generated by the set X of variables.
- $\mathbf{T}(X) = \langle T(X), \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ a term algebra (not an FL).
- Let $\mathbf{A}$ be an algebra of the type of FL. A valuation f on $\mathbf{A}$ is a homomorphism

$$f : \mathbf{T}(X) \longrightarrow \mathbf{A}.$$

- Given a set $E \cup \{s = t\}$ of equations, a class K of algebras,

$$E \models_{\mathsf{K}} s = t$$

iff for every $\mathbf{A} \in$ K and valuation f on $\mathbf{A}$, $f(u) = f(v)$, for all $(u = v) \in E$, implies $f(s) = f(t)$.

A class V of algebras (of the same type) is a variety if V $= HSP($V$)$:

- H: homomorphic images
- S: subalgebras
- P: direct products

Theorem (Birkhoff)

V is a variety iff V is equationally definable.

Some equations:

(e)	$xy = yx$	(commutativity)
(i)	$x \leq 1$	(integrality)
(c)	$x \leq xx$	(contractivity)
(dn)	$\neg\neg x \leq x$	(involutivity)
(pl)	$1 \leq (x \to y) \vee (y \to x)$	(prelinearity)
(div)	$x \wedge y = x(x \to y)$	(divisibility)

Full Lambek Calculus $\mathbf{FL}$

- The base system for substructural logics (Ono 90)
  $\approx$ Intuitionistic logic without structural rules.
- Formulas $=$ terms $T(X)$ of FL-algebras
- Sequents: $\Gamma \Rightarrow \Pi$
  (Γ: sequence of formulas, Π: at most one formula)
- Intuition:

$$\begin{array}{rcl} \alpha_1, \ldots, \alpha_n \Rightarrow \beta & \approx & \alpha_1 \cdots \alpha_n \leq \beta \\ \Rightarrow \beta & \approx & 1 \leq \beta \\ \alpha_1, \ldots, \alpha_n \Rightarrow & \approx & \alpha_1 \cdots \alpha_n \leq 0 \end{array}$$

Inference Rules of FL

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, A, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \Pi}\ \mathit{Cut} \qquad \frac{}{A \Rightarrow A}\ \mathit{Identity}$$

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow \Pi \quad \Gamma_1, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \vee B, \Gamma_2 \Rightarrow \Pi}\ \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \vee r \qquad \frac{}{\bot, \Gamma \Rightarrow \Pi}\ \bot l$$

$$\frac{\Gamma_1, A_i, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A_1 \wedge A_2, \Gamma_2 \Rightarrow \Pi}\ \wedge l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge r \qquad \frac{}{\Gamma \Rightarrow \top}\ \top r$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \cdot B, \Gamma_2 \Rightarrow \Pi}\ \cdot l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}\ \cdot r$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, B, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, A \backslash B, \Delta_2 \Rightarrow \Pi}\ \backslash l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \backslash B}\ \backslash r$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, B, \Delta_2 \Rightarrow \Pi}{\Delta_1, B/A, \Gamma, \Delta_2 \Rightarrow \Pi}\ /l \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}\ /r$$

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Pi}\ 1l \qquad \frac{}{\Rightarrow 1}\ 1r \qquad \frac{}{0 \Rightarrow}\ 0l \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ 0r$$

- Given a set $\Phi \cup \{s\}$ of sequents, $\Phi \vdash_{\mathbf{FL}} s$ if s is derivable from Φ by the inference rules of $\mathbf{FL}$.
- We often identify a formula φ with a sequent $\Rightarrow \varphi$.
- A substructural logic is an axiomatic extension of $\mathbf{FL}$, namely a set Φ of formulas closed under substitution and deduction: $\Phi \vdash_{\mathbf{FL}} \varphi \Longrightarrow \varphi \in \Phi$.

Algebraization Theorem

$\vdash_{\mathbf{FL}}$ corresponds to $\models_{\mathsf{FL}}$;

$$\Phi \vdash_{\mathbf{FL}} \psi \quad \text{iff} \quad \{1 \leq \varphi : \varphi \in \Phi\} \models_{\mathsf{FL}} 1 \leq \psi.$$

The substructural logics are in 1-1 correspondence with the subvarieties of FL.

Some axioms:

(e)	$(\varphi \cdot \psi) \to (\psi \cdot \varphi)$	(exchange)
(w)	$\varphi \to 1, \quad 0 \to \varphi$	(weakening)
(c)	$\varphi \to \varphi \cdot \varphi$	(contraction)
(dn)	$\neg\neg\varphi \to \varphi$	(double negation)
(pl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	(prelinearity)
(div)	$\varphi \wedge \psi \leftrightarrow \varphi \cdot (\varphi \to \psi)$	(divisibility)

(We informally write $\to$ for $\backslash, /$.)

Let $\{\mathbf{A}_i\}_{i\in I}$ be a family of algebras of the same type.
$\mathbf{A}$ is a subdirect product of $\{\mathbf{A}_i\}_{i\in I}$ if there is an embedding

$$e : \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$$

which is a surjection on each coordinate $\mathbf{A}_i$.

Proposition

Let $\{\theta_i\}_{i\in I}$ be congruences on $\mathbf{A}$ such that $\bigcap \theta_i = \triangle$ (diagonal). Then

$$e : \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}/\theta_i$$

Any subdirect product is of this form (up to isomorphism).

$\mathbf{A}$ is subdirectly irreducible (s.i.) if $\mathbf{A} \cong \mathbf{A}_i$ for some $i \in I$ whenever $\mathbf{A}$ is a subdirect product of $\{\mathbf{A}_i\}_{i \in I}$.

Theorem

Every algebra is a subdirect product of s.i. algebras.

$$\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i.$$

Corollary

Every variety is ISP-generated by its s.i. members.

$$\mathsf{V} = ISP(\mathsf{V}_{SI}).$$

Hence $E \models_{\mathsf{V}} t = u$ iff $E \models_{\mathsf{V}_{SI}} t = u$.

Facts

- Among Boolean algebras, $\mathbf{2}$ is the only s.i.
- An FLew-algebra $\mathbf{A}$ is s.i. iff it has the second greatest element.

Proposition

- If $\mathbf{A}$ is an s.i. FLew-algebra, then

$$a \vee b = 1 \iff a = 1 \text{ or } b = 1.$$

- Any s.i. MTL-algebra (prelinear FLew-algebra) is a chain.

- An MTL-algebra $\mathbf{A}$ is standard if $\mathbf{A} = \langle [0,1], min, max, *, \rightarrow, 1, 0 \rangle$.
- A standard algebra is completely determined by the t-norm $*$.

Theorem

The variety MTL is HSP-generated by standard algebras.

Proposition

Let $\mathbf{A}$ be a standard MTL algebra.

- $* : [0,1]^2 \longrightarrow [0,1]$ is left continuous.
- $\mathbf{A}$ is divisible $\mathbf{A} \models x \wedge y = x(x \rightarrow y)$ iff $*$ is continuous.

Theorem

The variety BL (of divisible MTL algebras) is HSP-generated by standard algebras in which $*$ is continuous.

Łukasiewicz t-norm:

$$x *_{Ł} y = max(x + y - 1, 0), \qquad x \to_{Ł} y = min(1 - x + y, 1)$$

Gödel t-norm:

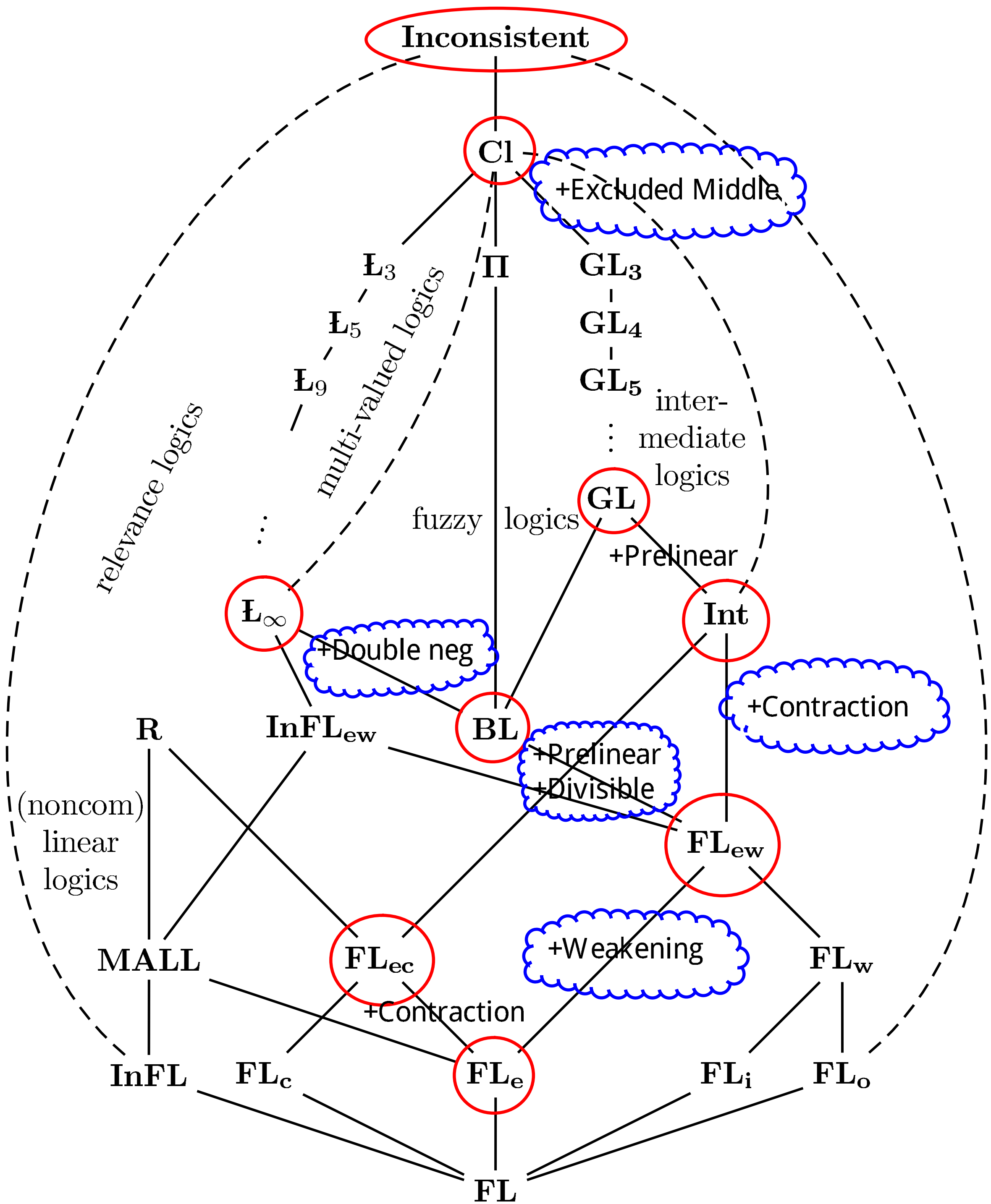
$$x *_{G} y = min(x, y), \qquad x \to_{G} y = \begin{cases} 1 & \text{if } x \leq y; \\ y & \text{if } x > y. \end{cases}$$

Product t-norm:

$$x *_{\Pi} y = xy, \qquad x \to_{\Pi} y = \begin{cases} 1 & \text{if } x \leq y; \\ y/x & \text{if } x > y. \end{cases}$$

Theorem

1. Ł $(\mathbf{BL} + (dn))$ is complete w.r.t. $*_{Ł}$.
2. $\mathbf{G}$ $(\mathbf{BL} + (c))$ is complete w.r.t. $*_{G}$.
3. Π $(\mathbf{BL}+??)$ is complete w.r.t. $*_{\Pi}$.



From
N. Galatos, P. Jipsen,
T. Kowalski and H. Ono,

Residuated Lattices:
An Algebraic Glimpse
at Substructural Logics, 2007.

Questions to be asked in SL

Let $\mathbf{L}$ be a consistent substructural logic and consider the decision problem:

$$\text{Given } \varphi, \quad \vdash_{\mathbf{L}} \varphi?$$

Theorem

1. Any $\mathbf{L}$ is coNP-hard.
2. If $\mathbf{L}$ is tabular ($\mathbf{L} = \mathbf{L}(\mathbf{A})$ with A finite), then $\mathbf{L}$ is coNP-complete.
3. If $\mathbf{L}$ satisfies the disjunction property, then $\mathbf{L}$ is PSPACE-hard.

2. and 3. are not necessary conditions. Nevertheless, coNP and PSPACE seem a natural way to classify logics into "semantically simple" and "computationally expressive" ones.

Let $\mathbf{L}$ be a consistent substructural logic and consider the decision problem:

$$\text{Given } \varphi, \quad \vdash_{\mathbf{L}} \varphi?$$

Theorem

1. Any $\mathbf{L}$ is coNP-hard.
2. If $\mathbf{L}$ is tabular ($\mathbf{L} = \mathbf{L}(\mathbf{A})$ with A finite), then $\mathbf{L}$ is coNP-complete.
3. If $\mathbf{L}$ satisfies the disjunction property, then $\mathbf{L}$ is PSPACE-hard.

2. and 3. are not necessary conditions. Nevertheless, coNP and PSPACE seem a natural way to classify logics into "semantically simple" and "computationally expressive" ones.

Dichotomy Problem

Is there a substructural logic which is neither coNP-complete nor PSPACE-hard?

$\mu\mathbf{L}$: Enrich the syntax with binder $\mu\alpha$ (α a propositional variable) and add an axiom for each $\varphi = \varphi(\alpha)$:

$$\mu\alpha.\varphi \leftrightarrow \varphi(\mu\alpha.\varphi).$$

Eg. letting $\varphi_0 := \mu\alpha.\neg\alpha$, we have $\varphi_0 \leftrightarrow \neg\varphi_0$.

Theorem

1. For any logic $\mathbf{L}$ above $\mathbf{FLc}$, $\mu\mathbf{L}$ is inconsistent.
2. For any logic $\mathbf{L}$ below Ł, $\mu\mathbf{L}$ is consistent.

Consistency with fixpoints

Theorem

1. For any logic $\mathbf{L}$ above $\mathbf{FLc}$, $\mu\mathbf{L}$ is inconsistent.
2. For any logic $\mathbf{L}$ below Ł, $\mu\mathbf{L}$ is consistent.

1.

$$\dfrac{[\varphi_0] \quad \dfrac{[\varphi_0]}{\neg\varphi_0}}{\dfrac{\bot}{\neg\varphi_0}}$$

2. It is sufficient to find a valuation v on $[0,1]$ that satisfies

$$\begin{array}{rcl} \alpha_1 & = & \varphi_1(\alpha_1,\ldots,\alpha_n) \\ & \vdots & \\ \alpha_n & = & \varphi_n(\alpha_1,\ldots,\alpha_n) \end{array}$$

In Ł, every formula denotes a continuous function. Hence such a v can be found by Brouwer's fixed point theorem.

In naive set theory one defines $r := \{x : x \notin x\}$ and obtains $r \in r \leftrightarrow r \notin r$. Nevertheless such a set theory can be consistent.

Open Problem (cf. White 79)

Is the naive set theory over Ł (or $\mathbf{MTL}$) consistent?

Herbrand's theorem

Predicate substructural logics

Let $\mathbf{L}$ be a propositional substructural logic.
$\mathbf{QL}$:= the predicate extension of $\mathbf{L}$ obtained by adding

$$\forall x.\alpha(x) \to \alpha(t) \qquad \alpha(t) \to \exists x.\alpha(x)$$

$$\frac{\beta \to \alpha(x)}{\beta \to \forall x.\alpha(x)} \qquad \frac{\alpha(x) \to \beta}{\exists x.\alpha(x) \to \beta} \qquad (x \text{ not free in } \beta)$$

Let $\mathbf{A}$ be a complete FL algebra.
An FL structure over $\mathbf{A}$ is $\mathcal{M} = (M, ^*)$ where

- M is a nonempty set
- * is an assignment:

$$f^* : M^n \longrightarrow M \quad \text{for each } n\text{-ary function symbol } f$$
$$p^* : M^n \longrightarrow A \quad \text{for each } n\text{-ary predicate symbol } p$$

Let $\mathbf{A}$ be a complete FL algebra.
An FL structure over $\mathbf{A}$ is $\mathcal{M} = (M, {}^*)$ where

- M is a nonempty set
- * is an assignment:

$$\begin{array}{ll} f^* : M^n \longrightarrow M & \text{for each } n\text{-ary function symbol } f \\ p^* : M^n \longrightarrow A & \text{for each } n\text{-ary predicate symbol } p \end{array}$$

Each formula $\varphi(\overline{a})$ with parameters $\overline{a} = a_1, \dots, a_n \in M$ is interpreted by $\varphi^*(\overline{a}) \in A$:

$$\begin{array}{rcl} (\forall x.\varphi(x, \overline{a}))^* & := & \bigwedge_{b \in M} \varphi^*(b, \overline{a}), \\ (\exists x.\varphi(x, \overline{a}))^* & := & \bigvee_{b \in M} \varphi^*(b, \overline{a}). \end{array}$$

Let $\mathbf{A}$ be a complete FL algebra.
An FL structure over $\mathbf{A}$ is $\mathcal{M} = (M, ^*)$ where

- M is a nonempty set
- * is an assignment:

$$f^* : M^n \longrightarrow M \quad \text{for each } n\text{-ary function symbol } f$$
$$p^* : M^n \longrightarrow A \quad \text{for each } n\text{-ary predicate symbol } p$$

Each formula $\varphi(\overline{a})$ with parameters $\overline{a} = a_1, \ldots, a_n \in M$ is interpreted by $\varphi^*(\overline{a}) \in A$:

$$(\forall x.\varphi(x, \overline{a}))^* \quad := \quad \bigwedge_{b \in M} \varphi^*(b, \overline{a}),$$
$$(\exists x.\varphi(x, \overline{a}))^* \quad := \quad \bigvee_{b \in M} \varphi^*(b, \overline{a}).$$

$$\mathcal{M} \models \varphi(\overline{a}) \quad \Longleftrightarrow \quad 1 \leq \varphi^*(\overline{a}).$$

Given a variety V of FL algebras,

$$\mathsf{QV} := \text{the class of complete algebras in V.}$$

$$\models_{\mathsf{QV}} \psi \iff \forall \mathbf{A} \in \mathsf{QV}.\ \forall \mathcal{M} \text{ over } \mathbf{A}.\quad \mathcal{M} \models \psi.$$

Soundness of Interpretation

Given a variety V of FL algebras,

$$\mathsf{QV} := \text{the class of complete algebras in V}.$$

$$\models_{\mathsf{QV}} \psi \iff \forall \mathbf{A} \in \mathsf{QV}.\ \forall \mathcal{M} \text{ over } \mathbf{A}.\quad \mathcal{M} \models \psi.$$

Soundness Theorem

Let $\mathbf{L}$ be a propositional substructural logic, $\mathsf{V} = \mathsf{V}(\mathbf{L})$, and $\Phi \cup \{\varphi\}$ a set of closed predicate formulas.

$$\Phi \vdash_{\mathbf{QL}} \varphi \implies \Phi \models_{\mathsf{QV}} \varphi.$$

Soundness of Interpretation

Given a variety V of FL algebras,

$$\mathsf{QV} := \text{the class of complete algebras in V.}$$

$$\models_{\mathsf{QV}} \psi \iff \forall \mathbf{A} \in \mathsf{QV}.\ \forall \mathcal{M} \text{ over } \mathbf{A}.\quad \mathcal{M} \models \psi.$$

Soundness Theorem

Let $\mathbf{L}$ be a propositional substructural logic, $\mathsf{V} = \mathsf{V}(\mathbf{L})$, and $\Phi \cup \{\varphi\}$ a set of closed predicate formulas.

$$\Phi \vdash_{\mathbf{QL}} \varphi \implies \Phi \models_{\mathsf{QV}} \varphi.$$

The converse direction, algebraic completeness, does not necessarily holds.
Proving it requires completions.

Let $\mathbf{A}$ be an FL algebra. A completion of $\mathbf{A}$ is a pair of a complete FL algebra $\mathbf{B}$ and an embedding $e : \mathbf{A} \hookrightarrow \mathbf{B}$.

We may assume $\mathbf{A} \subseteq \mathbf{B}$.

Completions

Let $\mathbf{A}$ be an FL algebra. A completion of $\mathbf{A}$ is a pair of a complete FL algebra $\mathbf{B}$ and an embedding $e : \mathbf{A} \hookrightarrow \mathbf{B}$.

We may assume $\mathbf{A} \subseteq \mathbf{B}$.

We consider 3 types of completion:

- MacNeille completions
  [Dedekind, MacNeille, Schmidt, Banaschewski . . .]
- Canonical extensions
  [Tarski, Jónson, Gehrke, Harding . . .]
- Hypercanonical extensions

MacNeille completions

$([0,1]_{\mathbb{Q}}, \min, \max)$ can be embedded into $([0,1]_{\mathbb{R}}, \min, \max)$.

What is the distinctive feature of this completion?

$([0,1]_{\mathbb{Q}}, \min, \max)$ can be embedded into $([0,1]_{\mathbb{R}}, \min, \max)$.

What is the distinctive feature of this completion?

For every $x \in [0,1]_{\mathbb{R}}$,

$$x = \sup\{a \in [0,1]_{\mathbb{Q}} : a \leq x\} = \inf\{a \in [0,1]_{\mathbb{Q}} : a \geq x\}.$$

MacNeille completions

$([0,1]_{\mathbb{Q}}, \min, \max)$ can be embedded into $([0,1]_{\mathbb{R}}, \min, \max)$.

What is the distinctive feature of this completion?

For every $x \in [0,1]_{\mathbb{R}}$,

$$x = \sup\{a \in [0,1]_{\mathbb{Q}} : a \leq x\} = \inf\{a \in [0,1]_{\mathbb{Q}} : a \geq x\}.$$

Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- join-dense if for every $x \in B$, $x = \bigvee\{a \in A : a \leq x\}$.
- meet-dense if for every $x \in B$, $x = \bigwedge\{a \in A : a \geq x\}$.

$([0,1]_{\mathbb{Q}}, \min, \max)$ can be embedded into $([0,1]_{\mathbb{R}}, \min, \max)$.

What is the distinctive feature of this completion?

For every $x \in [0,1]_{\mathbb{R}}$,

$$x = \sup\{a \in [0,1]_{\mathbb{Q}} : a \leq x\} = \inf\{a \in [0,1]_{\mathbb{Q}} : a \geq x\}.$$

Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- join-dense if for every $x \in B$, $x = \bigvee\{a \in A : a \leq x\}$.
- meet-dense if for every $x \in B$, $x = \bigwedge\{a \in A : a \geq x\}$.

Theorem (Schmidt 56, Banaschewski 56)

Every lattice $\mathbf{A}$ has a join-dense and meet-dense completion $\overline{\mathbf{A}}$ unique up to isomorphism, called the MacNeille completion.

MacNeille completions

Corollary

Every FL algebra $\mathbf{A}$ has a join-dense and meet-dense completion $\overline{\mathbf{A}}$, called the MacNeille completion.

MacNeille completion is regular (preserves all existing joins and meets). Hence it is useful for proving algebraic completeness (Ono 94, 12, etc.).

MacNeille completion is deeply connected to cut elimination in sequent calculus (Ciabattoni-Galatos-T. 12).

Canonical extensions

Let $\mathbf{C}$ be a Boolean algebra and $X_{\mathbf{C}}$ be its Stone space. Then

$$\begin{array}{lcll} \mathbf{C}^\sigma & := & (\mathcal{P}(X_{\mathbf{C}}), \cap, \cup, {}^C) & \\ e(a) & := & \{p : a \in p\} & : \mathbf{C} \longrightarrow \mathbf{C}^\sigma \end{array}$$

is a completion of $\mathbf{C}$.

Canonical extensions

Let $\mathbf{C}$ be a Boolean algebra and $X_{\mathbf{C}}$ be its Stone space. Then

$$\begin{array}{rcll} \mathbf{C}^\sigma & := & (\mathcal{P}(X_{\mathbf{C}}), \cap, \cup, {}^C) & \\ e(a) & := & \{p : a \in p\} & : \mathbf{C} \longrightarrow \mathbf{C}^\sigma \end{array}$$

is a completion of $\mathbf{C}$.

Let $\mathbf{D}$ be a bounded distributive lattice and $Y_{\mathbf{D}}$ be its Priestly space. Then

$$\mathbf{D}^\sigma \quad := \quad (\mathcal{P}_{\downarrow}(Y_{\mathbf{D}}), \cap, \cup)$$

is a completion of $\mathbf{D}$.

Canonical extensions

Let $\mathbf{C}$ be a Boolean algebra and $X_{\mathbf{C}}$ be its Stone space. Then

$$\begin{array}{rcll} \mathbf{C}^\sigma & := & (\mathcal{P}(X_{\mathbf{C}}), \cap, \cup, {}^C) & \\ e(a) & := & \{p : a \in p\} & : \mathbf{C} \longrightarrow \mathbf{C}^\sigma \end{array}$$

is a completion of $\mathbf{C}$.

Let $\mathbf{D}$ be a bounded distributive lattice and $Y_{\mathbf{D}}$ be its Priestly space. Then

$$\mathbf{D}^\sigma \quad := \quad (\mathcal{P}_\downarrow(Y_{\mathbf{D}}), \cap, \cup)$$

is a completion of $\mathbf{D}$.

These completions are somehow "canonical" and

$$\frac{\mathbf{C}^\sigma}{\mathbf{C}} = \frac{\mathbf{D}^\sigma}{\mathbf{D}}.$$

Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- dense if for every $x \in B$, there exist $C_i, D_j \subseteq A$ $(i \in I, j \in J)$ such that

$$x = \bigvee_{i \in I} \bigwedge C_i = \bigwedge_{j \in J} \bigvee D_j.$$

- compact if for every $C, D \subseteq A$,

$$\bigwedge C \leq \bigvee D \Longrightarrow \bigwedge C_0 \leq \bigvee D_0$$

for some finite $C_0 \subseteq C$ and $D_0 \subseteq D$.

Canonical extensions

Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- dense if for every $x \in B$, there exist $C_i, D_j \subseteq A$ $(i \in I, j \in J)$ such that

$$x = \bigvee_{i \in I} \bigwedge C_i = \bigwedge_{j \in J} \bigvee D_j.$$

- compact if

$$X = \bigcup \mathcal{O} \Longrightarrow X = \bigcup \mathcal{O}_0$$

for some finite $\mathcal{O}_0 \subseteq \mathcal{O}$.

Let $\mathbf{A}$ be a lattice. Its completion $\mathbf{B}$ is

- dense if for every $x \in B$, there exist $C_i, D_j \subseteq A$ $(i \in I, j \in J)$ such that

$$x = \bigvee_{i \in I} \bigwedge C_i = \bigwedge_{j \in J} \bigvee D_j.$$

- compact if for every $C, D \subseteq A$,

$$\bigwedge C \leq \bigvee D \Longrightarrow \bigwedge C_0 \leq \bigvee D_0$$

for some finite $C_0 \subseteq C$ and $D_0 \subseteq D$.

Theorem (Gehrke-Harding 01)

Every lattice $\mathbf{A}$ has a unique dense and compact completion $\mathbf{A}^\sigma$, called the canonical extension.

Corollary

Every FL algebra $\mathbf{A}$ has a dense and compact completion $\mathbf{A}^\sigma$, called the canonical extension.

The canonical extension of $([0,1]_\mathbb{Q}, \min, \max)$ is $(X, \cap, \cup)$ with

$$\begin{aligned} X \quad := \quad & \{[0,r] \ : \ r \in [0,1]_\mathbb{R}\} \cup \\ & \{[0,r) \ : \ r \in [0,1]_\mathbb{R}\}. \end{aligned}$$

Definition (Herbrand Property)

Let $\mathbf{L}$ be a propositional substructural logic. $\mathbf{QL}$ satisfies the Herbrand property if for every set Φ of universal formulas and every quantifier-free formula $\varphi(x)$,

$$\Phi \vdash_{\mathbf{QL}} \exists x.\varphi(x) \iff \Phi \vdash_{\mathbf{QL}} \varphi(t_1) \vee \cdots \vee \varphi(t_n)$$

for some $t_1, \ldots, t_n$.

Definition (Herbrand Property)

Let $\mathbf{L}$ be a propositional substructural logic. $\mathbf{QL}$ satisfies the Herbrand property if for every set Φ of universal formulas and every quantifier-free formula $\varphi(x)$,

$$\Phi \vdash_{\mathbf{QL}} \exists x.\varphi(x) \iff \Phi \vdash_{\mathbf{QL}} \varphi(t_1) \vee \cdots \vee \varphi(t_n) \text{ for some } t_1, \ldots, t_n.$$

Theorem

If V($\mathbf{L}$) is closed under compact completions, then $\mathbf{QL}$ satisfies the Herbrand property.

Theorem

If $\mathsf{V} = \mathsf{V}(\mathbf{L})$ is closed under compact completions, then $\mathbf{QL}$ satisfies the Herbrand property.

Proof: Let $\mathbf{A}$ be the Lindenbaum algebra of $\mathbf{QL}(\Phi)$ ($\mathbf{A} \in \mathsf{V}$).

Herbrand property via compact completions

Theorem

If $\mathsf{V} = \mathsf{V}(\mathbf{L})$ is closed under compact completions, then $\mathbf{QL}$ satisfies the Herbrand property.

Proof: Let $\mathbf{A}$ be the Lindenbaum algebra of $\mathbf{QL}(\Phi)$ $(\mathbf{A} \in \mathsf{V})$. It has a compact completion $\mathbf{B}$ in V.

Herbrand property via compact completions

Theorem

If $\mathsf{V} = \mathsf{V}(\mathbf{L})$ is closed under compact completions, then $\mathbf{QL}$ satisfies the Herbrand property.

Proof: Let $\mathbf{A}$ be the Lindenbaum algebra of $\mathbf{QL}(\Phi)$ $(\mathbf{A} \in \mathsf{V})$.
It has a compact completion $\mathbf{B}$ in V.
Let $\mathcal{M}$ be the term model over $\mathbf{B}$. Then $\mathcal{M} \models \Phi$.

Theorem

If V $=$ V$(\mathbf{L})$ is closed under compact completions, then $\mathbf{QL}$ satisfies the Herbrand property.

Proof: Let $\mathbf{A}$ be the Lindenbaum algebra of $\mathbf{QL}(\Phi)$ ($\mathbf{A} \in$ V).
It has a compact completion $\mathbf{B}$ in V.
Let $\mathcal{M}$ be the term model over $\mathbf{B}$. Then $\mathcal{M} \models \Phi$.
By soundness-compactness-completeness,

$$
\begin{aligned}
\Phi \vdash_{\mathbf{QL}} \exists x.\varphi(x) &\implies \mathcal{M} \models \exists x.\varphi(x) \\
&\implies 1 \leq \textstyle\bigvee_{t \in Term} \varphi^*(t) \text{ in } \mathbf{B} \\
&\implies 1 \leq_{\mathbf{B}} \varphi^*(t_1) \vee \cdots \vee \varphi^*(t_n) \text{ for some } \vec{t} \\
&\implies 1 \leq_{\mathbf{A}} \varphi^*(t_1) \vee \cdots \vee \varphi^*(t_n) \text{ for some } \vec{t} \\
&\implies \Phi \vdash_{\mathbf{QL}} \varphi(t_1) \vee \cdots \vee \varphi(t_n) \text{ for some } \vec{t}
\end{aligned}
$$

Theorem (Gehrke-Harding 01)

Let V be a variety of monotone lattice expansions.
If V is generated by finitely many finite algebras, then V is closed under canonical extensions.

Herbrand property for finite-valued logics

Theorem (Gehrke-Harding 01)

Let V be a variety of monotone lattice expansions.
If V is generated by finitely many finite algebras, then V is closed under canonical extensions.

Corollary

Let $\mathbf{L}$ be a finite lattice-valued logic with monotone/antimonotone operations. Then $\mathbf{QL}$ satisfies the Herbrand property.

Herbrand property for finite-valued logics

Theorem (Gehrke-Harding 01)

Let V be a variety of monotone lattice expansions.
If V is generated by finitely many finite algebras, then V is closed under canonical extensions.

Corollary

Let $\mathbf{L}$ be a finite lattice-valued logic with monotone/antimonotone operations. Then $\mathbf{QL}$ satisfies the Herbrand property.

It applies to all finite-valued modal/substructural logics (eg. finite-valued Łukasiewicz/superintuitionistic/relevant logics).

Theorem (Gehrke-Harding 01)

Let V be a variety of monotone lattice expansions.
If V is generated by finitely many finite algebras, then V is closed under canonical extensions.

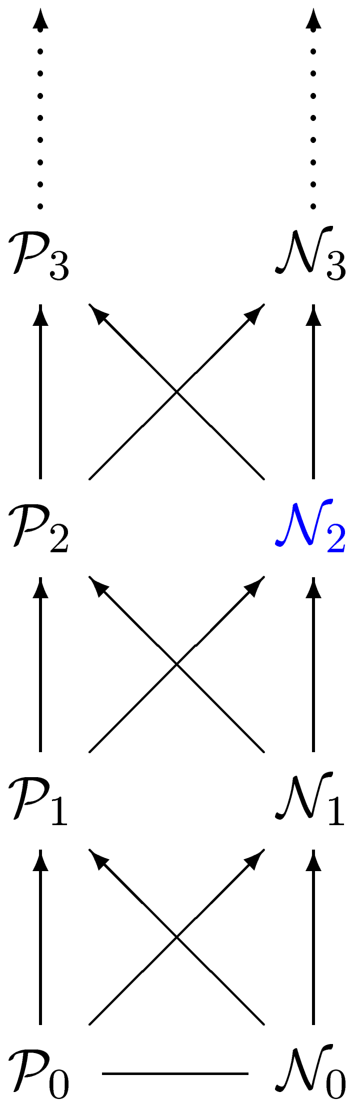
Corollary

Let $\mathbf{L}$ be a finite lattice-valued logic with monotone/antimonotone operations. Then $\mathbf{QL}$ satisfies the Herbrand property.

It applies to all finite-valued modal/substructural logics (eg. finite-valued Łukasiewicz/superintuitionistic/relevant logics).

The GH theorem is an algebraic counterpart of the uniform midsequent theorem for finite-valued logics (Baaz-Fermüller-Zach 94).

Substructural hierarchy



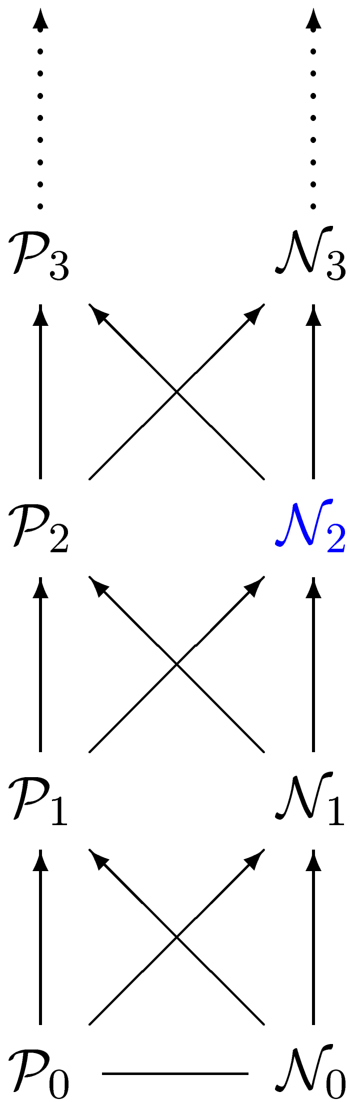
Classification of axioms

$$
\begin{aligned}
\mathcal{P}_0, \mathcal{N}_0 &::= \text{the set of variables} \\
\mathcal{P}_n &::= \mathcal{N}_{n-1} \mid 1 \mid \mathcal{P}_n \vee \mathcal{P}_n \mid \mathcal{P}_n \cdot \mathcal{P}_n \\
\mathcal{N}_n &::= \mathcal{P}_{n-1} \mid 0 \mid \mathcal{N}_n \wedge \mathcal{N}_n \mid \mathcal{P}_n \backslash \mathcal{N}_n \mid \mathcal{N}_n / \mathcal{P}_n
\end{aligned}
$$

Some $\mathcal{N}_2$ axioms:

$\alpha \to 1,\ 0 \to \alpha$	weakening
$\alpha \to \alpha \cdot \alpha$	contraction
$\alpha \cdot \alpha \to \alpha$	expansion
$\alpha^n \to \alpha^m$	knotted axioms $(n, m \geq 0)$
$\neg(\alpha \wedge \neg\alpha)$	no-contradiction

$\mathcal{N}_2$ corresponds to sequent calculus and MacNeille completions



Theorem (Ciabattoni-Galatos-T. 12)

1. Every $\mathcal{N}_2$ axiom can be transformed into a set of structural rules in sequent calculus.

2. For every set E of $\mathcal{N}_2$ axioms, the following are equivalent.

- $\mathbf{FL}(E)$ admits a sequent calculus with "strong" cut elimination.
- $\mathsf{V}(\mathbf{FL}(E))$ is closed under MacNeille completions.
- E is acyclic (a syntactic criterion).

Theorem (Gehrke-Harding-Venema 05)

Let V be a variety of monotone lattice expansions. If V is closed under MacNeille completions, it is also closed under canonical extensions.

Theorem (Gehrke-Harding-Venema 05)

Let V be a variety of monotone lattice expansions. If V is closed under MacNeille completions, it is also closed under canonical extensions.

MacNeille completions preserve involutivity: $\neg\neg\alpha \leftrightarrow \alpha$.

Canonical extensions preserve distributivity:
$(\alpha \vee \beta) \wedge \gamma \leftrightarrow (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

Theorem (Gehrke-Harding-Venema 05)

Let V be a variety of monotone lattice expansions. If V is closed under MacNeille completions, it is also closed under canonical extensions.

MacNeille completions preserve involutivity: $\neg\neg\alpha \leftrightarrow \alpha$.

Canonical extensions preserve distributivity: $(\alpha \vee \beta) \wedge \gamma \leftrightarrow (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

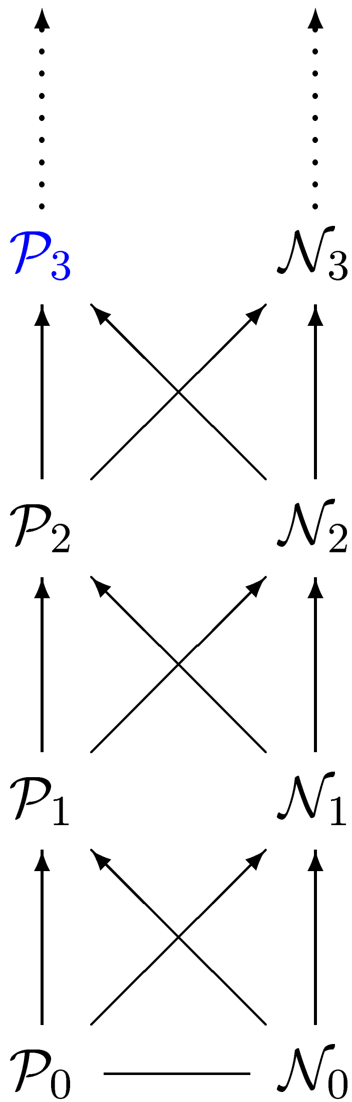
Corollary

Let $\mathbf{L}$ be a substructural logic axiomatized by

- □ acyclic $\mathcal{N}_2$ axioms
- □ and/or involutivity, distributivity

Then $\mathbf{QL}$ satisfies the Herbrand property.

Let's climb up the hierarchy



There is no reason to believe that canonical extensions preserve prelinearity.

We want completions that preserve $\mathcal{P}_3$ axioms:

$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\alpha \vee \neg\alpha$	excluded middle
$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$	weak nilpotent minimum
$\bigvee_{i=0}^{k}(\alpha_i \to \bigvee_{j \neq i} \alpha_j)$	bounded width $\leq k$
$\bigvee_{i=0}^{k}(\alpha_0 \wedge \cdots \wedge \alpha_{i-1} \to \alpha_i)$	bounded size $\leq k$

Idea from proof theory: making things 'hyper'

Sequents : $\Theta \quad \equiv \quad \alpha_1, \ldots, \alpha_n \Rightarrow \beta$

Idea from proof theory: making things 'hyper'

Sequents : $\Theta \quad \equiv \quad \alpha_1, \ldots, \alpha_n \Rightarrow \beta$

Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_m$, (meaning $\Theta_1 \vee \cdots \vee \Theta_m$)

Idea from proof theory: making things 'hyper'

Sequents : $\Theta \quad \equiv \quad \alpha_1, \ldots, \alpha_n \Rightarrow \beta$

Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_m, \quad$ (meaning $\Theta_1 \vee \cdots \vee \Theta_m$)

Prelinearity $(\alpha \to \beta) \vee (\beta \to \alpha)$ can be expressed by

$$\frac{\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda}{\Xi \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Lambda}$$

(Avron's communication rule)

Idea from proof theory: making things 'hyper'

Sequents : $\Theta \quad \equiv \quad \alpha_1, \ldots, \alpha_n \Rightarrow \beta$

Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_m$, (meaning $\Theta_1 \vee \cdots \vee \Theta_m$)

Prelinearity $(\alpha \to \beta) \vee (\beta \to \alpha)$ can be expressed by

$$\frac{\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda}{\Xi \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Lambda}$$

(Avron's communication rule)

Assuming exchange and weakening,

Theorem (Ciabattoni-Galatos-T.)

For every set E of $\mathcal{P}_3$ axioms,

- □ $\mathbf{FLew}(E)$ admits a hypersequent calculus with "strong" cut elimination.
- □ $\mathsf{V}(\mathbf{FLew}(E))$ is closed under hyperMacNeille completions.

Sequents : $\Theta \quad \equiv \quad \alpha_1, \ldots, \alpha_n \Rightarrow \beta$
Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_m$, (meaning $\Theta_1 \vee \cdots \vee \Theta_m$)
Prelinearity $(\alpha \to \beta) \vee (\beta \to \alpha)$ can be expressed by

$$\frac{\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda}{\Xi \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Lambda}$$

(Avron's communication rule)
Assuming exchange and weakening,

Theorem (Ciabattoni-Galatos-T.)

For every set E of $\mathcal{P}_3$ axioms,

- $\mathbf{FLew}(E)$ admits a hypersequent calculus with "strong" cut elimination.
- $\mathrm{V}(\mathbf{FLew}(E))$ is closed under hyperMacNeille completions.

We now make canonical extensions 'hyper.'

A residuated frame (Galatos-Jipsen) is $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!\!/, \varepsilon, \epsilon)$ such that

- $N \subseteq W \times W'$,
- $(W, \circ, \varepsilon)$ is a monoid, $\epsilon \in W'$,
- $x \circ y \; N \; z \iff x \; N \; z /\!\!/ y \iff y \; N \; x \backslash\!\backslash z$.

A residuated frame (Galatos-Jipsen) is $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \varepsilon, \epsilon)$ such that

- $N \subseteq W \times W'$,
- $(W, \circ, \varepsilon)$ is a monoid, $\epsilon \in W'$,
- $x \circ y \; N \; z \iff x \; N \; z /\!/ y \iff y \; N \; x \backslash\!\backslash z$.

Given $X \subseteq W$ and $Z \subseteq W'$,

$$X^{\triangleright} := \{z \in W' : x \; N \; z \text{ for every } x \in X\}$$
$$Z^{\triangleleft} := \{x \in W : x \; N \; z \text{ for every } z \in Z\}$$

A residuated frame (Galatos-Jipsen) is $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \varepsilon, \epsilon)$ such that

- $N \subseteq W \times W'$,
- $(W, \circ, \varepsilon)$ is a monoid, $\quad \epsilon \in W'$,
- $x \circ y \; N \; z \iff x \; N \; z /\!/ y \iff y \; N \; x \backslash\!\backslash z$.

Given $X \subseteq W$ and $Z \subseteq W'$,

$$\begin{aligned} X^{\triangleright} &:= \{z \in W' : x \; N \; z \text{ for every } x \in X\} \\ Z^{\triangleleft} &:= \{x \in W : x \; N \; z \text{ for every } z \in Z\} \end{aligned}$$

$(^{\triangleright}, {}^{\triangleleft})$ forms a Galois connection:

$$X \subseteq Z^{\triangleleft} \iff X^{\triangleright} \supseteq Z$$

that induces a closure operator $\gamma(X) := X^{\triangleright\triangleleft}$ on $\wp(W)$.

Given $\mathbf{W} = (W, W', N, \circ, \backslash\backslash, /\!/, \varepsilon, \epsilon)$ and $X, Y \subseteq W$,

$G(W) :=$ the set of Galois-closed subsets of W
$(X = \gamma(X) = X^{\triangleright\triangleleft})$

$$
\begin{array}{rcl}
X \backslash Y & := & \{y : x \circ y \in Y \text{ for every } x \in X\} \\
Y / X & := & \{y : y \circ x \in Y \text{ for every } x \in X\} \\
X \circ_\gamma Y & := & \gamma(X \circ Y) \\
X \cup_\gamma Y & := & \gamma(X \cup Y)
\end{array}
$$

Lemma

$$\mathbf{W}^+ := (G(W), \cap, \cup_\gamma, \circ_\gamma, \backslash, /, \varepsilon^{\triangleright\triangleleft}, \epsilon^{\triangleleft})$$

is a complete FL algebra, called the complex algebra of $\mathbf{W}$.

Let $\mathbf{A}$ be an FL algebra. By letting

$$\mathbf{W_A} := (A, A, \leq, \cdot, \backslash, /, 1, 0),$$

we obtain the MacNeille completion: $\mathbf{W}^+_\mathbf{A} = \overline{\mathbf{A}}$.

Let $\mathbf{A}$ be an FL algebra. By letting

$$\mathbf{W_A} := (A, A, \leq, \cdot, \backslash, /, 1, 0),$$

we obtain the MacNeille completion: $\mathbf{W_A^+} = \overline{\mathbf{A}}$.

By letting

$$\begin{array}{rcl} \mathbf{W_A^\sigma} & := & (\mathcal{F}_\mathbf{A}, \mathcal{I}_\mathbf{A}, N, \circ, \backslash\!\!\backslash, /\!\!/, \uparrow 1, \downarrow 0), \\ \mathcal{F}_\mathbf{A} & := & \text{the filters of } \mathbf{A} \\ \mathcal{I}_\mathbf{A} & := & \text{the ideals of } \mathbf{A} \\ f \ N \ i & := & f \cap i \neq \emptyset \\ f \backslash\!\!\backslash i & := & \{b \in A : \exists a \in f.\ ab \in i\} \\ i /\!\!/ f & := & \{b \in A : \exists a \in f.\ ba \in i\} \end{array}$$

we obtain the canonical extension: $\mathbf{W_A^{\sigma+}} = \mathbf{A}^\sigma$ (without recource to maximality/primality).

Let $\mathbf{A}$ be an FLew algebra. Let

$$\begin{aligned} \mathbf{W}^h_{\mathbf{A}} &:= (\mathcal{F}_{\mathbf{A}} \times \mathcal{I}_{\mathbf{A}}, \mathcal{I}_{\mathbf{A}} \times \mathcal{I}_{\mathbf{A}}, N, \circ, \backslash\!\!\backslash, /\!\!/, \varepsilon, \epsilon) \\ (f, j)\ N\ (i, k) &:= 1 \in (f \backslash\!\!\backslash i) \vee j \vee k \end{aligned}$$

It is a 'hyper' construction.

Theorem

$\mathbf{W}^h_{\mathbf{A}}$ is a residuated frame, so $\mathbf{W}^{h+}_{\mathbf{A}}$ is an FLew algebra, called the hypercanonical extension of $\mathbf{A}$.

Lemma

Hypercanonical extensions (applied to FLew algebras) are compact completions. They preserve all $\mathcal{P}_3$ axioms.

Lemma

Hypercanonical extensions (applied to FLew algebras) are compact completions. They preserve all $\mathcal{P}_3$ axioms.

Theorem

Let $\mathbf{L}$ be a substructural logic axiomatized by exchange, weakening and some $\mathcal{P}_3$ axioms. Then $\mathbf{QL}$ satisfies the Herbrand property.

Applies to $\mathbf{MTL}$, $\mathbf{G}$ (but not to Ł, $\mathbf{\Pi}$) and many more.

Can be generalized to extensions of $\mathbf{FL}$.

Proposition (Baaz-Metcalfe 08)

Ł does not satisfy the Herbrand property.

Proof: recall that for $a, b \in [0, 1]$

$$\begin{aligned} a \to_{Ł} b &= 1 && \text{if } a \leq b \\ &= 1 - a + b && \text{otherwise.} \end{aligned}$$

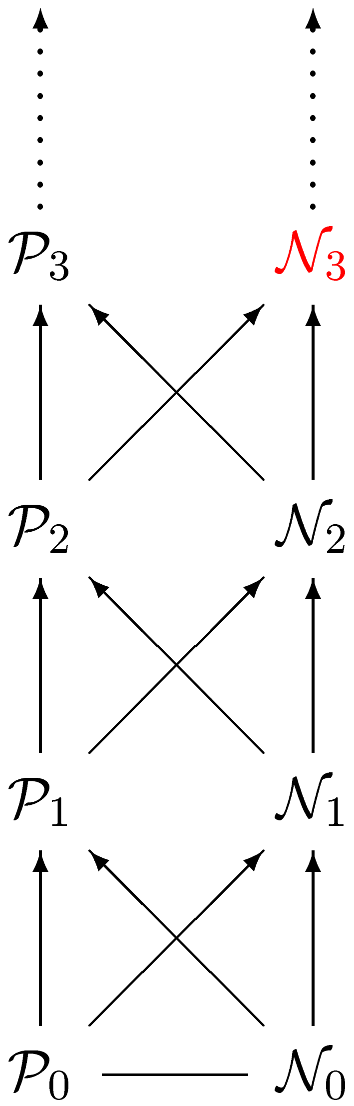
We have $\models_{Ł} \exists x.(p(fx) \to p(x))$ since

$$\sup_n [p(f^{n+1}c) \to p(f^n c)]^* = 1$$

under any valuation $*$. On the other hand, for any $N \in \mathbb{N}$ we can find $*$ such that

$$\bigvee_{n<N} [p(f^{n+1}c) \to p(f^n c)]^* < 1.$$

Failure of Herbrand's theorem



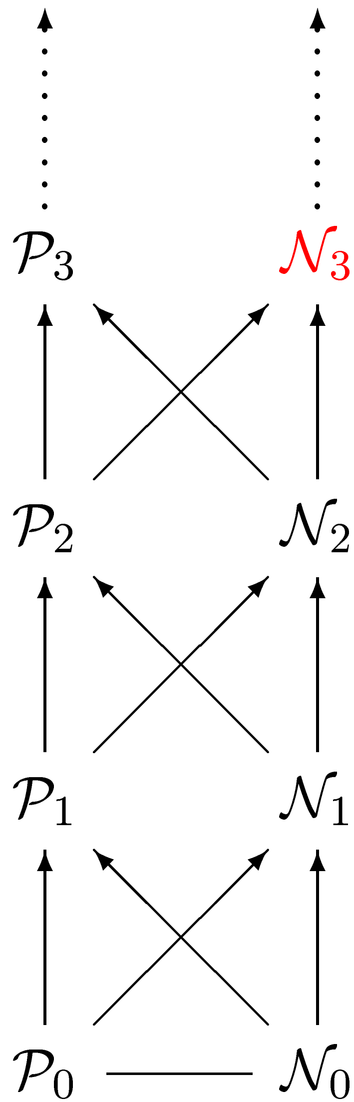
A deeper reason:

$$\text{(div)} \quad \varphi \wedge \psi \leftrightarrow \varphi \cdot (\varphi \to \psi) \qquad \in \mathcal{N}_3$$

Theorem (Kowalski-Litak 09)

The varieties BL and Ł are not closed under any completions.

Failure of Herbrand's theorem



A deeper reason:

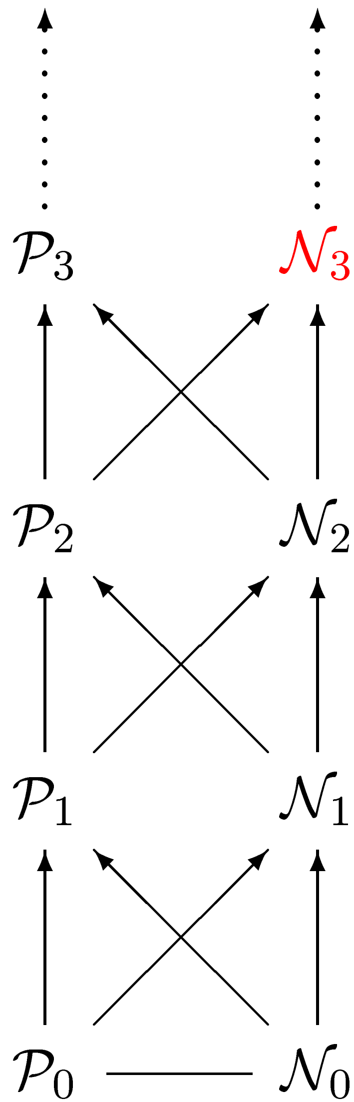
$$\text{(div)} \quad \varphi \wedge \psi \leftrightarrow \varphi \cdot (\varphi \to \psi) \qquad \in \mathcal{N}_3$$

Theorem (Kowalski-Litak 09)

The varieties BL and Ł are not closed under any completions.

Positive results hold uniformly below $\mathcal{P}_3$, but not above $\mathcal{N}_3$.

Failure of Herbrand's theorem



A deeper reason:

$$\text{(div)} \quad \varphi \wedge \psi \leftrightarrow \varphi \cdot (\varphi \to \psi) \qquad \in \mathcal{N}_3$$

Theorem (Kowalski-Litak 09)

The varieties BL and Ł are not closed under any completions.

Positive results hold uniformly below $\mathcal{P}_3$, but not above $\mathcal{N}_3$. Limitation of uniform proof theory!

General Herbrand's theorem

Herbrand's theorem is not only about ∃-formulas.

Herbrand's theorem is not only about ∃-formulas.

Herbrand's theorem for ∃∀-formulas:

$$\Phi \vdash \exists x \forall y . \varphi(x, y) \iff \Phi \vdash \varphi(t_1, y_1) \vee \cdots \vee \varphi(t_n, y_n)$$

where t_i does not contain $y_i, \ldots, y_n$.

Herbrand's theorem is not only about ∃-formulas.
Herbrand's theorem for ∃∀-formulas:

$$\Phi \vdash \exists x \forall y.\varphi(x, y) \iff \Phi \vdash \varphi(t_1, y_1) \vee \cdots \vee \varphi(t_n, y_n)$$

where t_i does not contain $y_i, \ldots, y_n$.

It can be further generalized to arbitrary prenex formulas. The general form requires the constant domain axiom (cd):

$$\forall x.(\alpha(x) \vee \beta) \leftrightarrow (\forall x.\alpha(x)) \vee \beta.$$

Its algebraic counterpart is meet complete distributivity:

$$\bigwedge_{i \in I}(x_i \vee y) = (\bigwedge_{i \in I} x_i) \vee y.$$

Lemma

Let $\mathbf{A}$ be an FL algebra.

- □ If $\mathbf{A}$ is distributive, then $\mathbf{A}^\sigma$ is completely distributive.
- □ If $\mathbf{A}$ is an MTL algebra, then $\mathbf{A}^h$ is completely distributive.

Lemma

Let $\mathbf{A}$ be an FL algebra.

- If $\mathbf{A}$ is distributive, then $\mathbf{A}^\sigma$ is completely distributive.
- If $\mathbf{A}$ is an MTL algebra, then $\mathbf{A}^h$ is completely distributive.

Theorem

Let $\mathbf{L}$ be a substructural logic. The general Herbrand's theorem holds for $\mathbf{QL}(cd)$ if

- *either* $\mathbf{L}$ is axiomatized by distributivity and some $\mathcal{N}_2$ axioms,
- *or* $\mathbf{L}$ is axiomatized by exchange, weakening, prelinearity and some $\mathcal{P}_3$ axioms.

We have explored the connection:

Closure under compact completions $\Longrightarrow$ Herbrand's theorem.

Conclusion

We have explored the connection:

Closure under compact completions $\Longrightarrow$ Herbrand's theorem.

Three open problems:

- ☐ Converse direction
- ☐ Abstract characterization of hypercanonical extensions
- ☐ 'Approximate' Herbrand's theorem (Baaz-Metcalfe 08)

Conclusion

We have explored the connection:

Closure under compact completions $\Longrightarrow$ Herbrand's theorem.

Three open problems:

- ☐ Converse direction
- ☐ Abstract characterization of hypercanonical extensions
- ☐ 'Approximate' Herbrand's theorem (Baaz-Metcalfe 08)

Hypercanonical extensions stem from a proof theoretic idea: making things 'hyper'.

Our long-span project

Explore the connection between proof theory and algebra in the context of nonclassical logics.