Recursion Theory and Fragments of Arithmetic

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Outline

Forward and Reverse Recursion Theory

- Reverse Recursion Theory is the “reverse” expression of Recursion Theory on weak fragments of arithmetic.
- Turing model of computation
- Many equivalent definitions including Σ^0_1 definability in arithmetic
- Modern Slogan: Recursion Theory studies definability.

Turing Functionals

- Turing reducibility $\leq_T$ and Turing degrees.
- We say $A \leq_T B$ if there is a Turing machine M such that with B as oracle M computes A.
- Definition: A Turing functional Φ is an r.e. set of triples of the form $\langle x, y, \sigma \rangle$ where $x, y \in \mathbb{N}$ and $\sigma \in \mathbb{N}^*$ satisfying monotonicity and consistency.
- One can further "rewrite" σ as a pair of finite sets P, N such that $P \cap N = \emptyset$.
- $A \leq_T B$ iff for some Turing functional Φ, $\Phi(B) = A$.

Degree Theory and Priority Methods

- Since 1944 Post's work, people focus mainly on degrees. Turing degrees and r.e. degrees (Both structures are complicated).
- Friedberg and Muchnik invented priority method to solve Post's problem, which asks if there is an intermediate r.e. degree, i.e., $0 < a < 0'$.
- Later Shoenfield and Sacks invented "infinite injury" method to show jump inversion theorems.
- In 1970's, Lachlan invented more complicated priority method.

Reverse Mathematics Motivation: Hilbert Program

- Hilbert Program: Justify “infinitary” math by “finitary” means.
- Program failed because of Gödel’s Theorems. But...
- Motivating question: Let’s find out the exact amount of “infinitary” tools needed.

Gödel Hierarchy

- Let T_1 and T_2 be theories. We say $T_1 < T_2$ iff T_2 proves the consistency of T_1.
- It turns out an almost linear hierarchy, quite robust (with some noise though).

Gödel Hierarchy

strong

- ⋮
- measurable cardinal
- ⋮
- ZFC
- ⋮

medium

- $\mathbb{Z}_2$
- ⋮
- Π^1_1-CA
- ATR_0
- ACA_0

weak

- WKL_0
- RCA_0
- ⋮
- bounded arithmetic
- ⋮

Goal of Reverse Mathematics

- Goal: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?
- To achieve this goal, we have to discover new proofs.

Second Order Arithmetic $\mathbb{Z}_2$

- Two sorted language: (first order part) Numbers, $+$, $\times$; (second order part) Sets; $\in$.
- Most of "standard mathematics" can be done in $\mathbb{Z}_2$.

Subsystems of $\mathbb{Z}_2$ - The Big Five

Basic axioms and

- RCA_0: Σ_1-induction and Δ_0-comprehension for $\varphi \in \Delta_0$, $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$.
- WKL_0: RCA_0 and every infinite binary tree has an infinite path. (essentially compactness)
- ACA_0: RCA_0 and for φ arithmetic, $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$.
- ATR_0:
- Π^1_1-CA_0.

A closer look at RCA_0

- A set is called *decidable* or *recursive* or *computable* if there is an algorithm which decides its membership. E.g. the set of all prime numbers.
- Models of RCA_0: Closure under $\leq_T$ and Turing join.
- In the (minimal) world RCA_0, only recursive sets exist.
- RCA_0 is the place to do constructive/finitary mathematics.

Recent Developments

- (old results) Simpson's book (2009) about classical math theorems and their correspondence with big five.
- (Beyond the Big Five): Mummert and Simpson 2005 provide an example of reverse mathematics at the level of Π^1_2-CA. The results are in the area of general topology.
- More and more exceptions (chaos around Ramsey's Theorem).

Motivations of Reverse Recursion Theory

- Under the influence of Reverse Mathematics, around 1980's, Groszek and Slaman studied "Reverse Recursion Theory".
- They work in *first order* Peano arithmetic and use the amount of induction to measure the complexity of recursion theoretic theorems.
- Recall: In Recursion Theory, the constructions are verified by induction; in particular, in priority arguments.
- Another motivation: Studying computability in more general domains, like in α-recursion theory.

Fragments of Peano Arithmetic

- We always assume the language has exponential function and satisfies $PA^- + B\Sigma_1$.
- Let $I\Sigma_n$ denote the induction schema for Σ^0_n-formulas; and $B\Sigma_n$ denote the Bounding Principle for Σ^0_n formulas.
- (Kirby and Paris, 1977) $\cdots \Rightarrow I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow \ldots$
- (Slaman 2004) $I\Delta_n \Leftrightarrow B\Sigma_n$.

Codes and $\mathcal{M}$-finite sets

- Let $\mathcal{M}$ be a model of $PA^- + B\Sigma_1$. In $\mathcal{M}$ we can do basic arithmetic.
- For example, every $a \in \mathcal{M}$ has a unique binary expansion $a(0)a(1)\dots a(l-1)$.
- We say a codes a set X iff for all i, $i \in X$ iff $a(i) = 1$.
- If X is coded then we call X is $\mathcal{M}$-finite.

Recursion Theory on $\mathcal{M}$

- A set $A \subset M$ is r.e iff A is Σ^0_1-definable in $\mathcal{M}$ with parameters.
- A set A is recursive iff A and $M \setminus A$ are r.e.
- A Turing functional Φ in $\mathcal{M}$ is an r.e. set of quadruples $\langle x, y, P, N \rangle$ satisfying the monotone and consistency conditions as before.
- So we can study recursion theory on weak fragments of Peano arithmetic.

Some Results in Reverse Recursion

- Over $PA^- + B\Sigma_1$: $I\Sigma_1 \Leftrightarrow$ Existence of low r.e. sets $\Leftrightarrow$ Sacks Splitting Theorem
- Over $PA^- + B\Sigma_2$: $I\Sigma_2 \Leftrightarrow$ Existence of high r.e. sets $\Leftrightarrow$ Minimal Pair Theorem
- As in Reverse Mathematics, new proofs are required when working in fragments.

Application one: Classifying Theorems

- It is difficult to classify priority methods, because of the different ways to label requirements.
- Reverse recursion offers an intrinsic measure of complexity of theorems.
- Example: Over $B\Sigma_2$, $I\Sigma_2 \Leftrightarrow$ the existence of maximal sets $\Leftrightarrow$ the existence of Friedberg numbering.
- As in Reverse Mathematics, people are exploring both higher and wider areas.

Frank Plumpton Ramsey (1903 - 1930)

- Ramsey “was a British mathematician who, in addition to mathematics, made significant and precocious contributions in philosophy and economics before his death at the age of 26.”

Ramsey's Theorem (History)

- Ramsey's Theorem appeared in his 1930 paper *On a problem of formal logic*.
- "While this theorem is the work Ramsey is probably best remembered for, he only proved it in passing, as a minor lemma along the way to his true goal in the paper, solving a special case of the decision problem for first-order logic, ..."
- Today it is an entire branch of mathematics, known as *Ramsey theory*.

Ramsey's Theorem

Definition

For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A.

Theorem (Ramsey, 1930)

Suppose $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$. *Then there is an infinite set* $H \subseteq \mathbb{N}$ *which is* f-homogeneous, *i.e.,* f *is constant on* $[H]^n$.

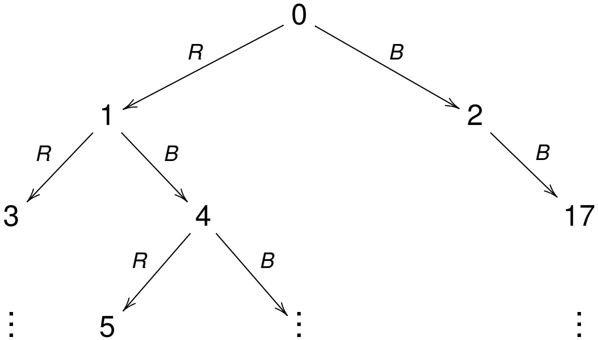
If we think of f as a k-coloring of the n-element subsets of $\mathbb{N}$, then all n-element subsets of H have the same color.

Informal reading: Within some sufficiently large systems, however disordered, there must be some order.

Sketch of a Proof for Pairs

Statement: If we colour pairs of natural numbers in two colors (Red and Blue), then there is an infinite subset $H \subset \mathbb{N}$, such that any pair formed by elements in H is coloured by the same colour.

Proof (idea). We enumerate a binary tree based on the colouring as illustrated by the following example:



Sketch of Proof (conti.)

We obtain an infinite binary tree. So there is an infinite branch of the tree. For example, the path $0 \to 1 \to 4 \to 5 \to \cdots$.

We can then read from the branch a homogeneous set by taking the "starting points of the red edges" (or blue).

Remarks

- We have used a version of Weak König Lemma.
- This tree is an "r.e. tree", so this version of Weak König Lemma is stronger than WKL_0.
- Applications in logic: Ramsey cardinals, indiscernibles, Paris and Harrington Theorem etc.

Ramsey's Theorem and Reverse Mathematics

- Motivating questions: What are the complexity of the homogenous set? What are the logical consequences of Ramsey's Theorem?
- After 40-year efforts of recursion theorists and reverse mathematicians, we now know:
  - ACA_0 is strictly stronger than RT^2_2, whereas WKL_0 is incomparable with RT^2_2.
  - RT^2_2 is strictly stronger than $B\Sigma_2$, but is strictly weaker than $I\Sigma_2$.
- Working on fragments helps.