Theories of concatenation, arithmetic, and undecidability

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Computability Theory and Foundations of Mathematics
Contents

- An introduction for Theories of Concatenation
- Weak theories of concatenation and arithmetic
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Back ground and known results

\[ C^2 \quad \triangledown \quad \triangledown \quad PA \]

\[ \triangledown \quad \triangledown \quad TC \quad \triangledown \quad Q \]
In A. Grzegorczyk’s paper “Undecidability without arithmetization” (2005), he defined a \((\circ, \varepsilon, \alpha, \beta)\)-theory \(TC\) of concatenation, whose axioms are:

(TC1) \(\forall x (x \circ \varepsilon = \varepsilon \circ x = x)\) Axiom for identity
(TC2) \(\forall x \forall y \forall z (x \circ (y \circ z) = (x \circ y) \circ z)\) Associativity
(TC3) Editors Axiom:
\[
\forall x \forall y \forall u \forall v (x \circ y = u \circ v \rightarrow \\
\exists w ((x \circ w = u \land y = w \circ v) \lor (x = u \circ w \land w \circ y = v)))
\]
(TC4) \(\alpha \neq \varepsilon \land \forall x \forall y (x \circ y = \alpha \rightarrow x = \varepsilon \lor y = \varepsilon)\)
(TC5) \(\beta \neq \varepsilon \land \forall x \forall y (x \circ y = \beta \rightarrow x = \varepsilon \lor y = \varepsilon)\)
(TC6) \(\alpha \neq \beta\)
About (TC3); editors axiom

If \( x \bowtie y = u \bowtie v \),
About (TC3); editors axiom

If \( x \bowtie y = u \bowtie v, \)
About (TC3); editors axiom

If $x \bowtie y = u \bowtie v$, 

\[
\begin{array}{c}
\text{Diagram with arrows for } x \bowtie y = u \bowtie v.
\end{array}
\]
TC : Theory of Concatenation

Definition

• \( x \sqsubseteq y \equiv \exists k \exists l ((k \bowtie x) \bowtie l = y) \)
• \( x \sqsubseteq_{\text{ini}} y \equiv \exists l (x \bowtie l = y) \)
• \( x \sqsubseteq_{\text{end}} y \equiv \exists k (k \bowtie x = y) \)
What can TC prove?

Proposition

TC proves the following assertions:

(1) $\forall x (x \alpha \neq \varepsilon \land \alpha x \neq \varepsilon)$

(2) $\forall x \forall y (xy = \varepsilon \rightarrow x = \varepsilon \land y = \varepsilon)$

(3) $\forall x \forall y (x \alpha = y \alpha \lor \alpha x = \alpha y \rightarrow x = y)$  *Weak cancellation*

Proposition

TC cannot prove the following assertions:

- $\forall x \forall y \forall z (xz = yz \rightarrow x = y)$  *cancellation*
TC and undecidability

Theorem [Grzegorczyk, 2005]

TC is undecidable.

Moreover,

Theorem [Grzegorczyk and Zdanowski, 2007]

TC is essentially undecidable.

Grzegorczyk and Zdanowski conjectured that
(i) TC and Q are mutually interpretable;
(ii) TC is minimal essentially undecidable theory.
Definition of interpretation

\( L_1, L_2 \) : languages of first order logic.

A relative translation \( \tau : L_1 \to L_2 \) is a pair \( \langle \delta, F \rangle \) such that

- \( \delta \) is an \( L_2 \)-formula with one free variable.
- \( F \) maps each relation-symbol \( R \) of \( L_1 \) to an \( L_2 \)-formula \( F(R) \).

We translate \( L_1 \)-formulas to \( L_2 \)-formulas as follows:

- \( (R(x_1, \cdots, x_n))^{\tau} := F(R)(x_1, \cdots, x_n) \);
- \( (\cdot)^{\tau} \) commutes with the propositional connectives;
- \( (\forall x \varphi(x))^{\tau} := \forall x (\delta(x) \to \varphi^{\tau}) \);
- \( (\exists x \varphi(x))^{\tau} := \exists x (\delta(x) \land \varphi^{\tau}) \).
Definition of interpretation

Definition (relative interpretation)

$L_1$-theory $T$ is (relatively) interpretable in $L_2$-theory $S$, denoted by $S \triangleright T$, iff there exists a relative translation $\tau : L_1 \rightarrow L_2$ such that

(i) $S \vdash \exists x \delta(x)$ and
(ii) for each axiom $\sigma$ of $T$, $S \vdash \sigma^\tau$.

Proposition

Let $S$ be a consistent theory. If $S \triangleright T$ and $T$ is essentially undecidable, then $S$ is also essentially undecidable.

The interpretability conserves the essential undecidability.
In 2009, the following results were proved by three ways independently: Visser and Sterken, Švejdar, and Ganea.

Theorem [2009]

TC interprets Q. (Hence TC ⪰ Q.)

Here, Q is Robinson’s arithmetic, whose language is \( (+, \cdot, 0, S) \)

\[
\begin{align*}
Q1) \ & \forall x \forall y (S(x) = S(y) \rightarrow x = y) \\
Q2) \ & \forall x (S(x) \neq 0) \\
Q3) \ & \forall x (x + 0 = x) \\
Q4) \ & \forall x \forall y (x + S(y) = S(x + y)) \\
Q5) \ & \forall x (x \cdot 0 = 0) \\
Q6) \ & \forall x \forall y (x \cdot S(y) = x \cdot y + x) \\
Q7) \ & \forall x (x \neq 0 \rightarrow \exists y (x = S(y)))
\end{align*}
\]

Q is essentially undecidable and finitely axiomatizable.
The theory $C^2$ of concatenation consists of $TC$ plus the following induction:

$$\varphi(\varepsilon) \land \forall x (\varphi(x) \rightarrow \varphi(x \cdot \alpha) \land \varphi(x \cdot \beta)) \rightarrow \forall x \varphi(x).$$

Here, $\varphi$ is a $(\cdot, \varepsilon, \alpha, \beta)$-formula.

Then, Ganea proved that

**Theorem [Ganea, 2009]**

$C^2$ and $PA$ are mutually interpretable.
Part I

A weak theory WTC of concatenation and mutual interpretability with $\mathbb{R}$
Arithmetic $\mathbb{R}$ (Mostowski-Robinson-Tarski, 1953)

$\langle +, \cdot, 0, 1, \leq \rangle$-theory $\mathbb{R}$

For each $n, m \in \omega$, ( $\bar{n}$ represents $1 + \cdots + 1$ )

(R1) $\bar{n} + \bar{m} = \bar{n + m}$

(R2) $\bar{n} \cdot \bar{m} = \bar{n \cdot m}$

(R3) $\bar{n} \neq \bar{m}$ (if $n \neq m$)

(R4) $\forall x (x \leq \bar{n} \rightarrow x = \bar{0} \lor x = \bar{1} \lor \cdots \lor x = \bar{n})$

(R5) $\forall x (x \leq \bar{n} \lor \bar{n} \leq x)$

* $\mathbb{R}$ is $\Sigma_1$-complete and essentially undecidable.

* $\mathbb{R} \not\models Q$, since $Q$ is finitely axiomatizable.
**Arithmetic $R_0$ (Cobham, 1960’s)**

$(+,\cdot, 0, 1, \leq)$-theory $R_0$

For each $n, m \in \omega$,

(R1) $\bar{n} + \bar{m} = n + m$

(R2) $\bar{n} \cdot \bar{m} = n \cdot m$

(R3) $\bar{n} \neq \overline{m}$ (if $n \neq m$)

(R4') $\forall x \left( x \leq \bar{n} \iff x = \bar{0} \lor x = \bar{1} \lor \cdots \lor x = \bar{n} \right)$

* $R_0$ interprets $R$ by translating ‘$\leq$’ by ‘$<$’ as follows:

$$x < y \equiv [0 \leq y \land \forall u \left( u \leq y \land u \neq y \rightarrow u + 1 \leq y \right)] \rightarrow x \leq y.$$  

* $R_0$ is **minimal** theory which is $\Sigma_1$-complete and essentially undecidable.
Arithmetic $\mathcal{R}_1$ (Jones and Shepherdson, 1983)

$(+,\cdot,0,1,\leq)$-theory $\mathcal{R}_1$

For each $n,m \in \omega$,

(R2) $\bar{n} \cdot \bar{m} = n \cdot m$
(R3) $\bar{n} \neq \bar{m}$ (if $n \neq m$)
(R4') $\forall x (x \leq \bar{n} \iff x = \bar{0} \lor x = \bar{1} \lor \cdots \lor x = \bar{n})$

$\mathcal{R}_1$ interprets $\mathcal{R}_0$ by J. Robinson’s definition of addition in terms of multiplication.

$\mathcal{R}_1$ is minimal theory which is essentially undecidable.
**WTC**: Weak Theory of Concatenation

$((\sqsubseteq, \varepsilon, \alpha, \beta)\text{-theory})$ \textit{WTC} has the following axioms: for each $u \in \{\alpha, \beta\}^*$,

(WTC1) $\forall x \sqsubseteq u (x \sqcup \varepsilon = \varepsilon \circ x = x)$;

(WTC2) $\forall x \forall y \forall z [(x \circ (y \circ z) \sqsubseteq u \lor (x \circ y) \circ z \sqsubseteq u] \rightarrow x \circ (y \circ z) = (x \circ y) \circ z$;

(WTC3) $\forall x \forall y \forall s \forall t [(x \circ y = s \circ t \land x \circ y \sqsubseteq u) \rightarrow \exists w ((x \circ w = s \land y = w \circ t) \lor (x = s \circ w \land w \circ y = t))];$

(WTC4) $\alpha \neq \varepsilon \land \forall x \forall y (x \circ y = \alpha \rightarrow x = \varepsilon \lor y = \varepsilon)$;

(WTC5) $\beta \neq \varepsilon \land \forall x \forall y (x \circ y = \beta \rightarrow x = \varepsilon \lor y = \varepsilon)$;

(WTC6) $\alpha \neq \beta$. 
**WTC: Weak Theory of Concatenation**

Here, \( \{\alpha, \beta\}^* \) is a set of finite strings over \( \{\alpha, \beta\} \), including empty string \( \varepsilon \). Let \( \{\alpha, \beta\}^+ := \{\alpha, \beta\}^* \setminus \{\varepsilon\} \).

For each \( u \in \{\alpha, \beta\}^* \), we represent \( u \) in theories as \( u \) by adding parentheses from *left*. For example, \( \alpha\alpha\beta\alpha = ((\alpha\alpha)\beta)\alpha \). We call each \( u \ (\in \{\alpha, \beta\}^*) \) *standard string*.

**Definition**

- \( x \sqsubseteq y \equiv (x = y) \lor \exists k \exists l [kx = y \lor xl = y \lor (kx)l = y \lor k(xl) = y] \)
- \( x \sqsubseteq_{\text{ini}} y \equiv (x = y) \lor \exists l (xl = y) \)
- \( x \sqsubseteq_{\text{end}} y \equiv (x = y) \lor \exists k (kx = y) \)
**Lemma**

\( \forall x \ (x \sqsubseteq u \leftrightarrow \bigvee_{v \sqsubseteq u} x = v) \).

**Theorem**

\( \Sigma_1 \)-completeness of \( \text{WTC} \)\n
\( \text{WTC} \) is \( \Sigma_1 \)-complete, that is, for each \( \Sigma_1 \)-sentence \( \varphi \), if \( \{ \alpha, \beta \}^\ast \models \varphi \) then \( \text{WTC} \vdash \varphi \).

\( \{ \alpha, \beta \}^\ast \) is a standard model of \( \text{TC} \).
WTC interprets \( R \)

From now on, we consider the translation of \( R \) into \( WTC \).

**translation of 0, 1, +**

We translate 0, 1, + as follows:

- \( 0 \Rightarrow \varepsilon \);
- \( 1 \Rightarrow \alpha \);
- \( x + y \Rightarrow x \triangleleft y \);
- \( x \leq y \Rightarrow \exists z (x \triangleleft z = y) \).

To translate the product, we have to make it **total on \( \omega \)**. To do this, we consider notion, “witness for product”.

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WTC interprets $\mathbb{R}$

An idea for the definition of witness

**Witness $w$ for $2 \times 3$ is as follows:**

\[
w = \beta \beta \beta \beta \alpha \beta \alpha \alpha \beta \alpha \alpha \beta (\alpha \alpha)(\alpha \alpha) \beta \beta \alpha \alpha \alpha \beta (\alpha \alpha)(\alpha \alpha)(\alpha \alpha) \beta \beta
\]

This is from the following interpretation of $2 \times 3$:

\[
(0,0) \rightarrow (1,2) \rightarrow (2,2+2) \rightarrow (3,2+2+2).
\]

That is, $2 \times 3$ is interpreted as **adding 2 three times**.

By the help of above idea, we can represent the relation “$w$ is a witness for product of $x$ and $y$” by a formula $\text{PWitn}(x, y, w)$. 
WTC interprets $R$

**Translation of product**

We translate the multiplication “$x \times y = z$” by
\[
\begin{align*}
(\exists w \text{Pwitn}(x, y, w) \land \beta \beta y \beta z \beta \beta \sqsubseteq_{\text{end}} w) \lor \\
(\neg (\exists w \text{Pwitn}(x, y, w))) \land z = 0.
\end{align*}
\]

**Lemma (uniqueness of the witness on $\omega$)**

For each $u, v \in \{\alpha\}^*$, there exists $w \in \{\alpha, \beta\}^*$ such that

WTC proves

$\text{Pwitn}(u, v, w) \land \forall w'(\text{Pwitn}(u, v, w') \rightarrow w = w').$

**Theorem**

WTC interprets $R$. 
Conversely, we can prove that $R$ interprets $WTC$, by applying the Visser’s following theorem:

Visser’s theorem (2009)

$T$ is interpretable in $R$ iff $T$ is locally finitely satisfiable

Here, a theory $T$ is **locally finitely satisfiable** iff any finite subtheory of $T$ has a finite model. Since $WTC$ is locally finitely satisfiable, we can get the following result:

Corollary

$R$ interprets $WTC$. 
Conclusion of part I

Theorem

WTC and \( R \) are mutually interpretable.

Corollary

(1) WTC is essentially undecidable.
(2) WTC interprets \( T \) iff \( T \) is locally finitely satisfiable.
(3) WTC cannot interpret TC.
(4) \( WTC_2 \) and \( WTC_n \) \( (n \geq 2) \) are mutually interpretable.

Here, \( WTC_n \) is WTC with \( n \)-th single-letters. (4) is from \( WTC_2 \triangleright R \triangleright WTC_n \triangleright WTC_2 \).
Part II

Minimal essential undecidability and variations of WTC
Minimal essential undecidability

**Question**

Is $\text{WTC}$ *minimal* essentially undecidable?

Here, *minimal* essentially undecidable means if one omits one axiom from $\text{WTC}$, then the resulting theory is no longer essentially undecidable. Again, $\text{WTC}$ is: for each $u \in \{\alpha, \beta\}$

(WTC1)  \( \forall x \sqsubseteq u (x \smile \varepsilon = \varepsilon \smile x = x) \);

(WTC2)  \( \forall x \forall y \forall z ([x \smile (y \smile z) \sqsubseteq u \lor (x \smile y) \smile z \sqsubseteq u] \rightarrow x \smile (y \smile z) = (x \smile y) \smile z) \);

(WTC3)  \( \forall x \forall y \forall s \forall t [(x \smile y = s \smile t \land x \smile y \sqsubseteq u) \rightarrow \exists w ((x \smile w = s \land y = w \smile t) \lor (x = s \smile w \land w \smile y = t))] \);

(WTC4)  \( \alpha \neq \varepsilon \land \forall x \forall y (x \smile y = \alpha \rightarrow x = \varepsilon \lor y = \varepsilon) \);

(WTC5)  \( \beta \neq \varepsilon \land \forall x \forall y (x \smile y = \beta \rightarrow x = \varepsilon \lor y = \varepsilon) \);

(WTC6)  \( \alpha \neq \beta \).
Minimal essential undecidability

Proposition

\[ \text{WTC}-(\text{WTC } k) \ (k = 3, 4, 5, 6) \text{ is not essentially undecidable.} \]

We can find a decidable consistent extension of each \( \text{WTC}-(\text{WTC } k) \) \( (k = 3, 4, 5, 6) \). Hence remaining question is \[ \text{WTC}-(\text{WTC } k) \ (k = 1, 2) \text{ is essentially undecidable?} \]
Minimal essential undecidability

**Proposition**

\[ \text{WTC} - (\text{WTC} k) \ (k = 3, 4, 5, 6) \text{ is not essentially undecidable.} \]

We can find a decidable consistent extension of each \( \text{WTC} - (\text{WTC} k) \) \((k = 3, 4, 5, 6)\). Hence remaining question is

\[ \text{WTC} - (\text{WTC} k) \ (k = 1, 2) \text{ is essentially undecidable?} \]

We have proved the following:

**Theorem (with O. Yoshida)**

\[ \text{WTC} - (\text{WTC} 1) \text{ can interpret WTC.} \]

Hence, \( \text{WTC} - (\text{WTC} 1) \) is still essentially undecidable.
This is proved by the following two lemmas.

**Lemma**

For each $u \in \{\alpha, \beta\}^*$, \(\text{WTC} - (\text{WTC1})\) proves $u \varepsilon = \varepsilon u = u$.

⇒ Without (WTC1), axiom for identity, we can prove that the empty string works well, as an identity element, for at least all standard strings.

**Lemma**

\(\text{WTC} - (\text{WTC1})\) ⊢ $\forall x (x \sqsubseteq u \land \exists x' (x = (\varepsilon x') \varepsilon) \rightarrow \bigvee_{v \sqsubseteq u} x = v)$.

Although we do not know whether \(\text{WTC} - (\text{WTC1})\) can prove $\forall x (x \sqsubseteq u \rightarrow \bigvee_{v \sqsubseteq u} x = v)$ or not, the above Lemma is strong enough to interpret \(\text{WTC} \) into \(\text{WTC} - (\text{WTC1})\).
Then, we interpret \( WTC \) in \( WTC - (WTC1) \) as follows:

**Domain** \( \delta(x) \equiv x = \alpha \lor \exists x' (x = (\beta x')\varepsilon) \).

*Remark that if \((\beta x')\varepsilon\) is standard, then \((\beta x')\varepsilon = \beta ((\varepsilon x')\varepsilon)\).*

**Constants** \( \varepsilon \Rightarrow \beta, \alpha \Rightarrow \beta \alpha, \beta \Rightarrow \beta \beta \).

**Let** \( \Omega(x, y) \equiv \exists! x' \exists! y' (x = (\beta x')\varepsilon \land y = (\beta y')\varepsilon) \).

Then we translate concatenation as \( \text{Conc}(x, y, z) \equiv \)

\[
x = \alpha \lor y = \alpha \rightarrow z = \alpha \\
\land \Omega(x, y) \rightarrow \exists x' \exists y' [x = (\beta x')\varepsilon \land y = (\beta y')\varepsilon \land z = (\beta ((x'\varepsilon)y'))\varepsilon] \\
\land \text{o.w.} \rightarrow z = \alpha.
\]

**Lemma**

For each \( w \in \{\alpha, \beta\}^* \), \( WTC - (WTC1) \) can prove that if \( \text{Conc}(x, y, \beta w) \), then \( x \) and \( y \) are also standard.
WTC − (WTC1) ⊳ ◇ WTC

Question

Is WTC − (WTC1) minimal essentially undecidable?
Question

Is $\text{WTC}-(\text{WTC1})$ minimal essentially undecidable?

Theorem (K. Higuchi)

$\text{WTC}-(\text{WTC1})$ is interpretable in $\text{S2S}$.

Here, $\text{S2S}$ is a monadic second-order logic whose language is $L = \{S_0, S_1, (P_a)_{a \in A}\}$. $S_0, S_1$ are two successors and $P_a$’s are unary predicates. Then, $\text{S2S} := \{\varphi \mid \varphi \text{ is an } L\text{-sentence} \& \{0, 1\}^* \models \varphi\}$. $\text{S2S}$ is proved to be decidable by M. O. Rabin (1969).

Theorem

$\text{WTC}-(\text{WTC1})$ is minimal essentially undecidable theory.
On the other hand, we can consider the theory of concatenation without empty string: $(\circ, \alpha, \beta)$-theory $\text{TC}^{-\varepsilon}$ has the following axioms:

(TC$^{-\varepsilon}1$) $\forall x \forall y \forall z (x \circ (y \circ z) = (x \circ y) \circ z)$  

Associativity

(TC$^{-\varepsilon}2$) Editors Axiom:

$$\forall x \forall y \forall s \forall t (x \circ y = s \circ t \rightarrow (x = s \land y = t) \lor$$

$$\exists w ((x \circ w = s \land y = w \circ t) \lor (x = s \circ w \land w \circ y = t)))$$

(TC$^{-\varepsilon}3$) $\forall x \forall y (\alpha \neq x \circ y)$

(TC$^{-\varepsilon}4$) $\forall x \forall y (\beta \neq x \circ y)$

(TC$^{-\varepsilon}5$) $\alpha \neq \beta$
A weak version $\text{WT}C^{-\varepsilon}$ of $\text{T}C^{-\varepsilon}$ has the following axioms:

For each $u \in \{\alpha, \beta\}^+$,

(WTC$^{-\varepsilon}$1) $\forall x \forall y \forall z [(x \leftarrow y \leftarrow z) \sqsubseteq u \lor (x \leftarrow y) \leftarrow z \sqsubseteq u]$
\[ \rightarrow x \leftarrow (y \leftarrow z) = (x \leftarrow y) \leftarrow z; \]

(WTC$^{-\varepsilon}$2) $\forall x \forall y \forall s \forall t [(x \leftarrow y = s \leftarrow t \land x \leftarrow y \sqsubseteq u) \rightarrow$
\[ (x = y) \land (s = t) \lor \]
\[ \exists w ((x \leftarrow w = s \land y = w \leftarrow t) \lor (x = s \leftarrow w \land w \leftarrow y = t))]; \]

(WTC$^{-\varepsilon}$3) $\forall x \forall y (x \leftarrow y \neq \alpha)$;

(WTC$^{-\varepsilon}$4) $\forall x \forall y (x \leftarrow y \neq \beta)$;

(WTC$^{-\varepsilon}$5) $\alpha \neq \beta$.

For this theory, we proved the following:
**Proposition**

\( WTC^{-\varepsilon} \) and \( WTC \) are mutually interpretable. Hence \( WTC^{-\varepsilon} \) is essentially undecidable.

\( WTC \triangleright WTC^{-\varepsilon} \) is easy. We interpret \( WTC \) in \( WTC^{-\varepsilon} \) as:

- **Domain** \( \delta(x) \equiv x = \alpha \lor x = \beta \lor \exists x' \ (x = \beta x') \).
- **Constants** \( \varepsilon \Rightarrow \beta, \alpha \Rightarrow \beta \alpha, \beta \Rightarrow \beta \beta \).
- \( x \bowtie y = z \) Let \( \Omega(x, y) \equiv \exists x' \exists y' \ (x = \beta x' \land y = \beta y') \), and translate the concatenation by \( \text{Conc}(x, y, z) \equiv \)

\[
[x = \alpha \lor y = \alpha \rightarrow z = \alpha] \land [x = \beta \rightarrow z = y] \land [y = \beta \rightarrow z = x] \land \\
[\Omega(x, y) \rightarrow \exists x' \exists y' \ (x = \beta x' \land y = \beta y' \land z = \beta (x'y'))] \land \\
[o.w. \rightarrow z = \alpha].
\]
\( WTC^{-\varepsilon} \) is minimal essentially undecidable

**Theorem**

\( WTC^{-\varepsilon} \) is **minimal** essentially undecidable.

This result partially contributes the following question by Grzegorczyk and Zdanowski:

**Question**

Is \( TC^{-\varepsilon} \) minimal essentially undecidable?

The remaining part of the question is the essential undecidability of \( TC^{-\varepsilon} - (TC^{-\varepsilon}1) \), that is, \( TC \) without associative law. We can easily find an decidable extension of each \( TC^{-\varepsilon} - (TC^{-\varepsilon}k) \), \((k = 2, 3, 4, 5)\).
Variations of WTC: WTC+(TC1) + (TC2) ⊬ ◊ WTC

Recall that

(TC1) ∀x (x ∈ ε = ε x = x)

(TC2) ∀x ∀y ∀z (x ⌢ y z = x ⌢ y z)

(TC3) ∀x ∀y ∀s ∀t [(x ⌢ y = s ⌢ t) →
                    ∃w ((x ⌢ w = s ∧ y = w ⌢ t) ∨ (x = s ⌢ w ∧ w ⌢ y = t))]

Proposition

WTC interprets WTC+(TC1) + (TC2)

Because WTC+(TC1) + (TC2) is locally finitely satisfiable.

Proposition

WTC can not interpret WTC+(TC3).

Because WTC+(TC3) is not locally finitely satisfiable.
Conclusion of Part II

The following are mutually interpretable ($n \geq 2$):

\begin{align*}
\text{WTC}_n + (\text{Identity}) + (\text{Assoc}) \\
\text{WTC}_n + (\text{Identity}) \\
\text{WTC}_n + (\text{Assoc})  & \quad \text{WTC}_n^{-\varepsilon} + (\text{Assoc}) \\
\text{WTC}_n & \quad \text{WTC}_n^{-\varepsilon} \\
\text{WTC}_n - (\text{WTC1})
\end{align*}

**Theorem**

\textbf{WTC} - (\textbf{WTC1}), \textbf{WTC}^{-\varepsilon} is minimal essentially undecidable.
Questions

(1) Is WTC-(Identity) $\Sigma_1$-complete ?
   $\Rightarrow$ Our conjecture is **NO**.

(2) WTC+ (Editors Axiom) $\vdash$ TC ?
   $\Rightarrow$ Our conjecture is **YES**.

(3) Are there some natural theory $T$ such that
   TC $\vdash T \vdash WTC$ and $WTC \not\vdash T$ and $T \not\vdash TC$ ?
References


WTC interprets R

**Definition of “Good”**

We define the formula $\text{Good}(x)$ as follows:

$$\text{Good}(x) \equiv \text{ID}(x) \land \text{AS}(x) \land \text{EA}(x),$$

where

- $\text{ID}(x) \equiv \forall s \sqsubseteq x(s \sqsubseteq \varepsilon = \varepsilon \sqsubseteq s = s)$;

- $\text{AS}(x) \equiv \forall s_0 \forall s_1 \forall s_2[(s_0 \sqsubseteq (s_1 \sqsubseteq s_2) \sqsubseteq x \lor (s_0 \sqsubseteq s_1) \sqsubseteq (s_0 \sqsubseteq s_1) \sqsubseteq s_2 \sqsubseteq x) \rightarrow s_0 \sqsubseteq (s_1 \sqsubseteq s_2) = (s_0 \sqsubseteq s_1) \sqsubseteq s_2]$

- $\text{EA}(x) \equiv \forall s_0 \forall s_1 \forall t_0 \forall t_1[(s_0 \sqsubseteq s_1 = t_0 \sqsubseteq t_1 \land s_0 \sqsubseteq s_1 \sqsubseteq x) \rightarrow \exists w\left((s_0 \sqsubseteq w = t_0 \sqsubseteq s_1 \sqsubseteq s_1 = w \sqsubseteq t_1) \lor (s_0 = t_0 \sqsubseteq w \land w \sqsubseteq s_1 = t_1)\right)]$
WTC interprets $\mathcal{R}$

Properties of Good

(1) For each $u \in \{\alpha, \beta, \gamma\}^*$, $\text{WTC} \vdash \text{Good}(u)$;

WTC proves the following assertions:

(2) $\forall x (\text{Good}(x) \rightarrow \forall y \subseteq x \text{Good}(y))$, that is
Good is closed under taking substrings.
To translate the product, we define “witness for product”.

First, we define a notion “number strings” as follows:

Definition of “\(\text{Num}\)”

We define the formula \(\text{Num}(x)\) as follows:

\[
\text{Num}(x) \equiv \forall y((y \subseteq x \land y \neq \varepsilon) \rightarrow \alpha \sqsubseteq_{\text{end}} y).
\]

Fact

For each \(u \in \{\alpha\}^*\), \(\text{WTC} \vdash \text{Num}(u)\).
We define a formula $\text{PWitn}(x, y, w)$ as follows:

(i) $\text{Num}(x) \land \text{Num}(y) \land \text{Good}(w)$;

(ii) $\beta y \beta \sqsubseteq_{\text{ini}} w$;

(iii) $\exists z (\text{Num}(z) \land \beta y z \beta \sqsubseteq_{\text{end}} w)$;

(iv) $\forall p \forall z (\text{Num}(z) \land p \beta y z \beta = w \rightarrow \forall z' (\text{Num}(z') \rightarrow \neg (\beta y z' \beta \sqsubseteq p \beta))$);

(v) $\forall p \forall q \forall s_2 \forall t_2 [(\text{Num}(s_2) \land \text{Num}(t_2) \land p \beta s_2 \gamma t_2 \beta q = w \land p \neq \varepsilon) \rightarrow (\exists s_1 \exists t_1 (\text{Num}(s_1) \land \text{Num}(t_1) \land s_2 = s_1 \alpha \land t_2 = t_1 x \land \beta s_1 \gamma t_1 \beta \sqsubseteq_{\text{end}} p \beta))]$;

(vi) $\forall p \forall q \forall s \forall t ((\text{Num}(s_1) \land \text{Num}(t_1) \land p \beta s \gamma t \beta q = w \land q \neq \varepsilon) \rightarrow \beta s \alpha \gamma t x \beta \sqsubseteq_{\text{ini}} \beta q)$. 


WTC interprets $R$

$\text{PWitn}(x, y, w)$

$w$
WTC interprets $R$

$\text{PWitn}(x, y, w)$

condition (ii)

$\beta \gamma \beta$

$\stackrel{\text{w}}{\beta \gamma \beta}$
WTC interprets $R$

$$\text{PWitn}(x, y, w)$$

condition (iii)

$$\beta y \gamma z \beta$$ for some $z \in W$
WTC interprets $R$

$\text{P Witn}(x, y, w)$

**condition (iv)**

$\beta y \gamma$ does not appear

$\beta y \gamma z \beta$
WTC interprets $R$

Translation of product

We translate the multiplication “$x \times y = z$” into the formula $M(x, y, z)$ as follows:

$$M(x, y, z) \equiv (\exists! w \text{PWitn}(x, y, w) \land \gamma z \beta \sqsubseteq_{\text{end}} w) \lor$$

$$\neg(\exists! w \text{PWitn}(x, y, w))) \land z = 0.$$
Main theorem

For each \( u, v \in \{a\}^+ \), there exists \( w \in \{a, b, c\}^+ \) such that \( \text{WTC proves} \)

\[
\text{PWitn}(u, v, w) \land \forall w' (\text{PWitn}(u, v, w') \rightarrow w = w').
\]

In what follows, we see the each steps of the proof of this main theorem.
WTC interprets $R$

**Lemma**

WTC proves the following assertions:

1. $\text{Good}(x\beta s) \land x\beta s = y\beta t \land \lnot(\beta \sqsubseteq s) \land \lnot(\beta \sqsubseteq t) \\
   \rightarrow (x = y \land s = t)$.

2. $\text{Good}(x\beta s\beta p) \land x\beta s\beta p = y\beta t\beta \land \lnot(\beta \sqsubseteq s) \land \lnot(\beta \sqsubseteq t)$
   
   $\rightarrow \begin{cases} 
   p \neq \epsilon \rightarrow \exists w(x\beta s\beta w = y\beta \land wt\beta = p) \lor \\
   p = \epsilon \rightarrow (x = y \land s = t). 
   \end{cases}$
WTC interprets $R$

If $x^\beta s^\beta p = y^\beta t^\beta$, 

(a) $p \neq \varepsilon$

(b) $p = \varepsilon$
WTC interprets $\mathbb{R}$

Existence of the witness

Fix $u \in \{a\}^+$. We can prove the existence of the witness $w \in \{a, b, c\}^+$ by the meta-induction on the length of $v \in \{a\}^+$. 
To prove the uniqueness of the witness, we prove this by the following two steps: Fix $u, v \in \{a\}^+$ and let $w \in \{a, b, c\}^+$ be some witness for $u, v$. In WTC, let $w'$ be such that $\text{PWitn}(u, v, w')$. Then,

**Step 1**

1. For each $k, l \in \{a\}^+$,
   \[ \text{WTC} \vdash \forall p (p \beta k \gamma l \beta \sqsubseteq_{\text{ini}} w \rightarrow p \beta k \gamma l \beta \sqsubseteq_{\text{ini}} w'); \]
2. $\text{WTC} \vdash w \sqsubseteq_{\text{ini}} w'$. 
WTC interprets $R$

$w \quad \beta \gamma \beta$

$w'$
WTC interprets $R$
WTC interprets $R$

$w$  

$w'$  

$\beta \alpha \gamma \mu \beta$
WTC interprets $R$

$w$ $\frac{\beta \alpha \gamma u \beta}{\beta \alpha \gamma u \beta}$

$w'$ $\beta \alpha \gamma u \beta$
WTC interprets $R$

$w$  \[ \beta \alpha \gamma \nu \mu \beta \]

$w'$
WTC interprets $R$

$w$

$w'$

$\beta \alpha \alpha \gamma \mu \nu \beta$

$\beta \alpha \alpha \gamma \mu \nu \beta$
WTC interprets R

\[ w \]

\[ w' \]
We prove this by way of contradiction. Let us assume that \( \exists q (wq = w' \land q \neq \varepsilon) \).
WTC \textbf{interprets} R

\textbf{Step 2}

WTC \vdash \overline{w} = w'.

We prove this by way of contradiction. Let us assume that \( \exists q (\overline{w}q = w' \land q \neq \varepsilon) \).
WTC interprets $R$

Step 2

\[
\text{WTC} \vdash w = w'.
\]

We prove this by way of contradiction. Let us assume that $\exists q (wq = w' \land q \neq \varepsilon)$.

\[
\begin{array}{ccc}
\text{w} & \beta \gamma \zeta_0 \beta \\
\hline
\text{w'}
\end{array}
\]
We prove this by way of contradiction. Let us assume that $\exists q (wq = w' \land q \neq \varepsilon)$.

$\vdash w = w'$.
**WTC interprets R**

**Step 2**

\[ \text{WTC} \vdash \overline{w} = \overline{w}'. \]

We prove this by way of contradiction. Let us assume that \( \exists q (\overline{w}q = \overline{w}' \land q \neq \varepsilon) \).

\[ \begin{array}{c}
\overline{w} \\
\beta \vee \gamma \overline{z}_0 \beta \\
\hline
\beta \vee \gamma \overline{z}_0 \beta \quad \beta \vee \gamma \overline{z}_1 \beta
\end{array} \]

\[ w \\
\overline{w}' \]
**WTC interprets** $R$

**Step 2**

$WTC \vdash \bar{w} = w'$.

We prove this by way of contradiction. Let us assume that $\exists q (\bar{w}q = w' \land q \neq \varepsilon)$.

\[ w \quad \text{contradict to the def. of PWitn} \quad \beta \nu \gamma z_0 \beta \]

\[ w' \quad \beta \nu \gamma z_1 \beta \]